

OPTIMAL PORTFOLIO OF THE GOVERNMENT PENSION INVESTMENT FUND BASED ON THE SYSTEMIC RISK EVALUATED BY A NEW ASYMMETRIC COPULA

BY YAN LIU

Waseda University

To evaluate systemic risk has been a central subject in economic and financial problems. One method to evaluate systemic risk is “CoVaR”, which represents the relationship of quantiles of the asset returns. In order to naturally use “CoVaR”, we construct an asymmetric copula such that conditional median of one random variable is not influenced by the other. We find that the empirical CoVaR of the Nikkei index evaluated by the new copula is more stable than the usual symmetric copula. We propose a new method to construct the optimal portfolio by considering CoVaR and numerical results for the GPIF are shown with our new asymmetric copula.

1. Introduction. After the financial crisis caused by the bankruptcy of Lehman Brothers in 2008 and 2009, the word “too big to fail” has spread all over the United States. The unprecedented financial crisis and recession made economists, financiers and econometricians consider a new measure of systemic risks of a financial firm in an entire financial system (Acharya, Engle and Richardson (2012)). One factor of the financial solvency of a financial firm is usually evaluated by its capital buffer, sometimes defined as

$$CB = E - k(D + E), \quad 0 < k < 1,$$

where E denotes the market value of its equity and D denotes the book value of its debt. If we assume that the systemic event debt cannot be renegotiated, then the equity, and thus the capital buffer is determined by the asset return over time. Therefore, the new measure of systemic risks is often defined in terms of the asset return of the certain financial firm.

One approach receiving notable attention, called “CoVaR”, has been proposed in Adrian and Brunnermeier (2011). Although VaR, the value at risk, has been widely used to evaluate the financial risk in various situations, it does not capture the latent systemic risks associated with any one other individual financial firm. The definition of “CoVaR” is given below.

DEFINITION 1.1. Let R_0 be the asset return. VaR_0 at level τ -quantile is defined by

$$\text{VaR}_0(\tau) = \inf\{x \in \mathbb{R}; P(R_0 \leq x) \geq \tau\}.$$

Suppose R_i and R_m denote respectively the asset returns of institution i and market. Then CoVaR_i conditional on institution i is defined by

$$(1.1) \quad \text{CoVaR}_i(\tau, \tau^*) = \inf\{x \in \mathbb{R}; \mathbb{P}(R_m \leq x \mid R_i = \text{VaR}_i(\tau^*)) \geq \tau\}.$$

To evaluate CoVaR , we have to know the joint distribution of the asset returns of institution i and market. One statistical approach is to use the copula to express it. The copula $C : [0, 1]^d \rightarrow [0, 1]$ is a function that decomposes the joint distribution to its marginals as

$$\mathbb{P}(X_1 \leq x_1, \dots, X_d \leq x_d) = C(F_{X_1}(x_1), \dots, F_{X_d}(x_d)),$$

where F_{X_1}, \dots, F_{X_d} are the marginal distribution of random variables X_1, \dots, X_d . The properties of copula function can be found in Nelsen (2007). We restrict our focus on the special case of $d = 2$ in this paper since we are interested in the evaluation of CoVaR .

Another intuition from statistics tells us to consider (1.1) in the sense of conditional quantile. A popular method to investigate the conditional quantile is the quantile regression, which is introduced in Koenker and Bassett (1978). As an alternative to ordinary least squares to determine the relationship between random variables, the method has been in the spotlight. From basics to applications, see Koenker (2005). For comprehensive approach to conditional quantile, Hua and Joe (2014) and Bernard and Czado (2015) are referred to. One attractive point of quantile-based inference is that it does not require that the random variables under consideration have finite moments. The treatment of random variables with infinite variance always puzzles the economists, financiers and econometricians. For this reason, the new concept CoVaR is appreciated on its own merits. Our goal is to find the optimal portfolio of the Government Pension Investment Fund (GPIF), whose assets under management amount to almost 140 trillion Yen. It is genuinely “too big to fail” financial system. At the same time, to evaluate its risk becomes the cause of worry for its management committee. We apply CoVaR to the evaluation of the systemic risk and furthermore construct the optimal portfolio for the GPIF. The unique idea in this paper is that we generated a new copula such that conditional median of one random variable is not influenced by the other random variable. Generally, this property does not hold when we only consider symmetric copulas. Further, the invariant conditional median seems more natural when we model financial returns. Our new method to construct the optimal portfolio of the GPIF by evaluating systemic risks is not trivial, so the change of the optimal portfolio according to the change of the corresponding quantile τ^* is not monotone, either. The numerical results could be referred to for the management of the systemic risk of the GPIF.

This paper is organized as follows. In Section 2, we consider the conditional quantile independence in general. In Section 3, we provide a new copula such that conditional median of one random variable is not affected by the other random variable. Empirical results for comparison of our asymmetric copula and usual symmetric copula by the Nikkei index are provided in Section 4. In Section 5, we present the new portfolio method with CoVaR and the optimal portfolio of the GPIF under the evaluation of CoVaR .

2. CoVaR independent of τ^* . In this section, we consider $\text{CoVaR}_i(\tau, \tau^*)$ independent of any τ^* . It is of some theoretical interest to study this special relationship between two continuous random variables X and Y , whose joint distribution is defined by the copula C . The concept can be formulated by

(i) the joint distribution of (X, Y) :

$$(2.1) \quad F_{X,Y}(x, y) = C(F_X(x), F_Y(y));$$

(ii) the quantile of Y independence of X :

$$(2.2) \quad Q_{Y|X}(\tau | X) = Q_Y(\tau) \quad \text{almost surely.}$$

From Sklar's Theorem, the existence of copula C in (2.1) is guaranteed and C is unique. Under this formulation, it is straightforward to see that the concept of CoVaR is

$$\text{CoVaR}_X(\tau, \tau^*) = Q_{Y|X}(\tau | X = Q_X(\tau^*)).$$

This concept stems from careful considerations on the dependence between the τ -quantile level of random variable Y and the τ^* -quantile level of random variable X . This is obviously different from the dependent relationships between two random variables in the mean level. Shao and Zhang (2014) considered the following three relationships between two random variables:

(i) The random variables X and Y are uncorrelated:

$$\text{Cov}(X, Y) = 0;$$

(ii) The conditional mean of Y given X is independent of X :

$$E(Y | X) = EY \quad \text{almost surely;}$$

(iii) The random variables X and Y are mutually independent:

$$X \perp Y.$$

It is not difficult to see that (iii) is the strongest concept among all three relationships. The order of these three concepts is (iii) \Rightarrow (ii) \Rightarrow (i). It has to be noted, however, that these three concepts are well-defined on the different spaces. While (iii) is defined by any real-valued random variables on probability space Ω , (ii) and (i) are well-defined on $L^1(\Omega)$ and $L^2(\Omega)$, respectively.

To well grasp the relationships between random variables X and Y , the scope is not supposed to be restricted to a specific subspace for random variables. This leads us to consider a moment-free relationship (2.2) at each quantile level τ^* and τ between random variables X and Y .

Let us consider the expression for the τ th quantile of the distribution of Y conditional on X . From (2.1), we have

$$C(u, v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)),$$

for $0 \leq u \leq 1$, $0 \leq v \leq 1$. Furthermore, it is not difficult to see that

$$(2.3) \quad \frac{\partial C(u, v)}{\partial u} = F_{Y|X}(F_Y^{-1}(v) | F_X^{-1}(u)) := C_*(v | u).$$

Thus, the τ -th quantile of the distribution of Y conditional on X is expressed by

$$(2.4) \quad Q_{Y|X}(\tau | F_X^{-1}(\tau^*)) = Q_Y(C_*^{-1}(\tau | \tau^*)).$$

Suppose $(U, V) = (F_X(X), F_Y(Y))$. Then (U, V) has uniformly distributed marginals on $[0, 1]$. From the equivalence of the σ -algebra generated by $F_X^{-1}(U)$ and X for continuous random variables, we can interpret the statement (2.2) as

$$C_*^{-1}(\tau | U) = \tau, \quad \text{almost surely,}$$

for uniformly distributed random variable U on $[0, 1]$.

Obviously, the conditional quantile independence (2.2) at any quantile level τ is equivalent to (iii) the independence between X and Y . The conditional quantile independence, however, at any specific quantile level τ is possible. The specification in quantile τ of Y given quantile τ^* of X makes us magnify the property of independence at a local area of the neighborhood of τ . We explain the former property in the following examples and leave the latter to the next section.

EXAMPLE 1. Suppose $C : [0, 1]^2 \rightarrow [0, 1]$ is a bivariate copula defined in each case.

(I) Independent copula $C(u, v) = uv$.

From the definition of $C_*(v, u)$, we have

$$C_*(v | u) = v.$$

For any τ th quantile of the distribution of Y conditional on X ,

$$C_*^{-1}(\tau | U) = \tau, \quad \text{almost surely.}$$

Therefore, we obtain

$$Q_{Y|X}(\tau | X) = F_Y^{-1}(\tau) = Q_Y(\tau).$$

In consequence, (iii) the independence between X and Y implies conditional quantile independence (2.2) at any τ -quantile.

Given $\tau = 0.01, 0.1, 0.5, 0.9, 0.99$ from below to above in Figure 1, we show the conditional quantile $Q_{Y|X}(\tau | X)$ when (X, Y) has standard normal distributed marginals at left hand side and (X, Y) has standard Cauchy distributed marginals at right hand side.

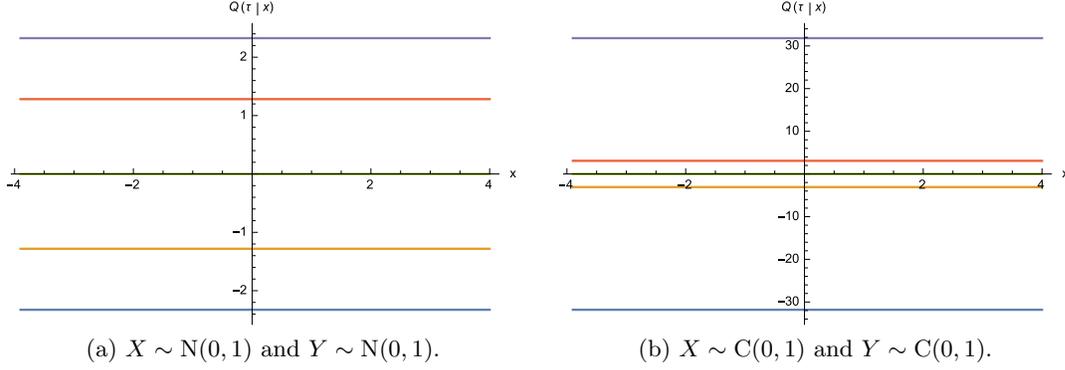


Fig 1: Conditional quantile $Q_{Y|X}(\tau | X = x)$ with different marginal distributions.

(II) Other well-known copula functions.

To fully understand the conditional quantile independence (2.2), we investigated the necessary and sufficient condition for each following well-known copula. The approach to consider the well-known copulas, however, is not helpful to find the situation that conditional quantile independence happens at some specific quantile but the full independence (iii) does not. To see the reason, we return back to equations (2.3)–(2.4). The existence of the conditional quantile τ requires that (2.3) is a function of v . On the other hand, we require that $Q_{Y|X}(\tau | X)$ in (2.4) is independent of X from the conditional quantile independence. As a result, $\partial C(u, v)/\partial u$ has to be a function of v but independent of u . This is impossible other than the case of the independent copula. The following well-known copulas from Bouyé and Salmon (2009) are some examples:

- For Ali-Mikhail-Haq copula

$$(2.5) \quad C(u, v; d) = \frac{uv}{1 - d(1-u)(1-v)}, \quad -1 \leq d < 1,$$

(2.2) holds if and only if $d = 0$.

- For Frank copula

$$C(u, v; d) = -\frac{1}{d} \log \left(1 + \frac{(\exp(-du) - 1)(\exp(-dv) - 1)}{\exp(-d) - 1} \right), \quad d \neq 0,$$

(2.2) holds if and only if $d \rightarrow 0$.

- For Farlie-Gumbel-Morgenstern copula

$$C(u, v; d) = uv \left(1 + d(1-u)(1-v) \right), \quad -1 \leq d \leq 1,$$

(2.2) holds if and only if $d = 0$.

- For Gumbel-Hougaard copula

$$C(u, v; d) = \exp \left(- \left[(-\log u)^d + (-\log v)^d \right]^{1/d} \right), \quad d \geq 1,$$

(2.2) holds if and only if $d = 1$.

- For Joe copula

$$C(u, v; d) = 1 - \left((1-u)^d + (1-v)^d - (1-u)^d(1-v)^d \right)^{1/d}, \quad d \geq 1,$$

(2.2) holds if and only if $d = 1$.

In all above examples, we omit the calculation processes since they are easy to check. It is seen that the condition for the parameter d makes the corresponding copula be the independent copula. From the heuristics points, we give the quantile dependence of standard normal marginals in Frank copula in Figure 2 with $d = 16$ at the left hand side and $d = 1$ at the right hand side. τ is defined as 0.01, 0.1, 0.5, 0.9, 0.99 from below to above. All these results match our previous explanation.

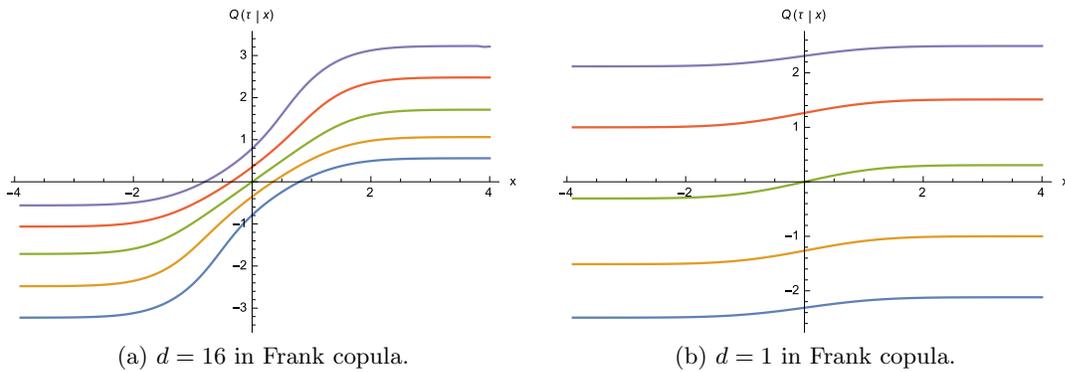


Fig 2: Conditional quantile $Q_{Y|X}(\tau | X = x)$ with different parameter d in Frank copula.

3. CoVaR locally independent of τ^* . In this section, we consider the example of local independence between the τ - and τ^* -quantile level of random variables Y and X . As what we have seen in the last section, the local independence between two different quantile levels does not happen if the copula function is symmetric except the trivial case. On the other hand, the study of asymmetric copula has still been undeveloped so far.

In the following, we show an example of asymmetric copula whose marginals are locally quantile independent at $\tau = \tau^* = 1/2$.

EXAMPLE 2 (A copula $C(u, v)$ satisfying $Q_{Y|X}(1/2 | X) = Q_Y(1/2)$ a.s.). To construct an asymmetric copula, we apply the method suggested in Wu (2014). Consider Ali-Mikhail-Haq copula (2.5). From Theorem 1 in Wu (2014),

$$\begin{aligned}\check{C}(u, v; d) &= C(u, 1; d) - C(u, 1 - v; d) \\ &= \frac{uv(1 + d(u - 1))}{1 + d(u - 1)v}\end{aligned}$$

is also a bivariate copula. Applying Lemma 4 in Wu (2014), we obtain a new copula

$$\begin{aligned}(3.1) \quad \hat{C}(u, v; d) &= \frac{1}{2}C(u, v; d) + \frac{1}{2}\check{C}(u, v; d) \\ &= \frac{uv}{2(1 - d(1 - u)(1 - v))} + \frac{1}{2}\left(u - \frac{u(1 - v)}{1 - d(1 - u)v}\right).\end{aligned}$$

This copula is obviously not the independent copula if $d \neq 0$. We exclude the case of $d = 0$. In reality, this copula (3.1) satisfies

$$(3.2) \quad Q_{Y|X}(1/2 | X) = Q_Y(1/2).$$

After some tedious calculations, we obtain

$$\begin{aligned}\frac{\partial \hat{C}(u, v)}{\partial u} &= \frac{v}{2(1 - d(1 - u)(1 - v))} - \frac{du(1 - v)v}{2(1 - d(1 - u)(1 - v))^2} \\ &\quad + \frac{1}{2}\left(1 + \frac{du(1 - v)v}{(1 - d(1 - u)v)^2} - \frac{1 - v}{1 - d(1 - u)v}\right).\end{aligned}$$

This is a function of u and v , so it guarantees the existence of the inverse function of τ . Furthermore, since

$$\tau = \frac{\partial \hat{C}(u, v)}{\partial u} \Big|_{v=1/2} = 1/2$$

is independent of u , (3.2) has been shown.

Now, let us observe the new copula defined by (3.1). Suppose (X, Y) are marginally distributed as standard normal. We generated the following scatter plots with the sample size 10^3 . As we mentioned in Example 2, when $d = 0$, it is independent copula (Figure 3).

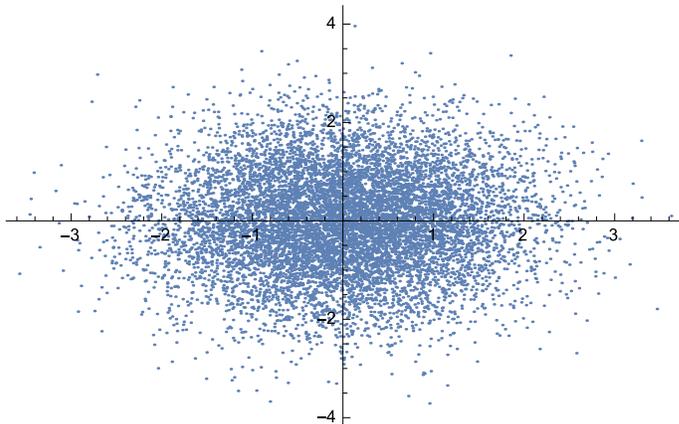
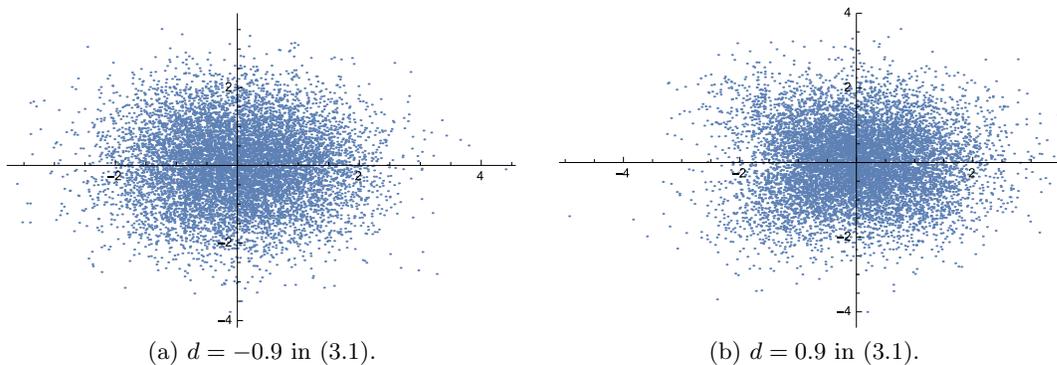


Fig 3: Independent copula with standard normal marginals.

On the contrary, we plotted (3.1) for the case of $d = -0.9$ and 0.9 in Figure 4. Obviously, the samples from the joint distribution of (X, Y) are not symmetric.



(a) $d = -0.9$ in (3.1).

(b) $d = 0.9$ in (3.1).

Fig 4: Scatter plot of new copula (3.1) with $d = -0.9$ (left) and $d = 0.9$ (right).

Next, we observe the changes in the conditional τ th quantile of the new copula conditional on $X = x$. τ is defined as 0.01, 0.1, 0.5, 0.9, 0.99 from below to above in Figure 5.

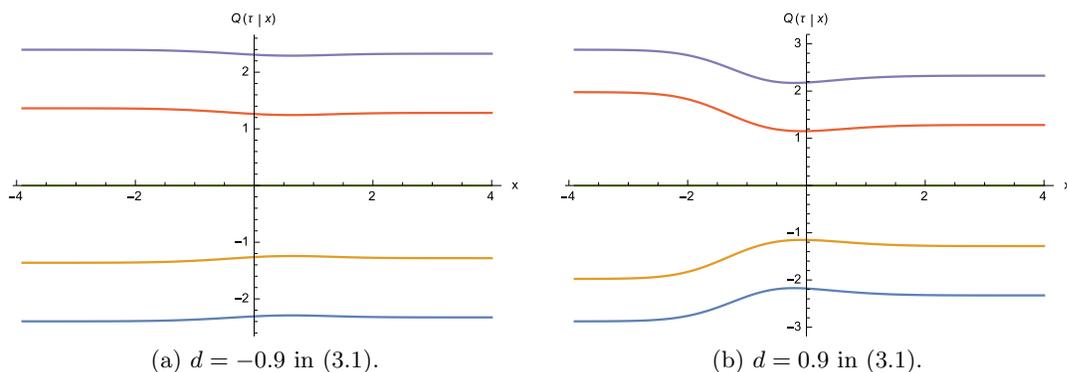


Fig 5: Conditional quantile of new copula (3.1) with $d = -0.9$ (left) and $d = 0.9$ (right).

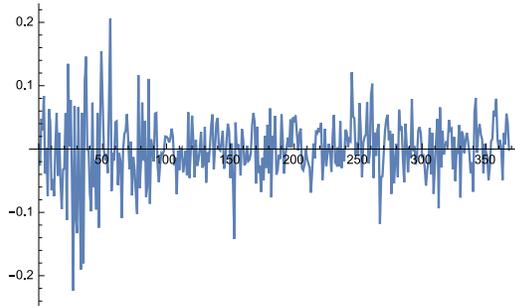
In either case above, the quantiles of both marginals X and Y are not symmetric. We see that the median of Y does not influenced by the change of the quantile of X , which is corresponding to what we have shown above.

4. Empirical results. In this section, we study the current situations of the Nikkei index from the point of view of systemic risks. We use the Gumbel-Hougaard copula and the new copula (3.1) as models for comparison. Our focus is on the CoVaR of the Nikkei index conditional on the asset returns of the following financial institutions (Table 1).

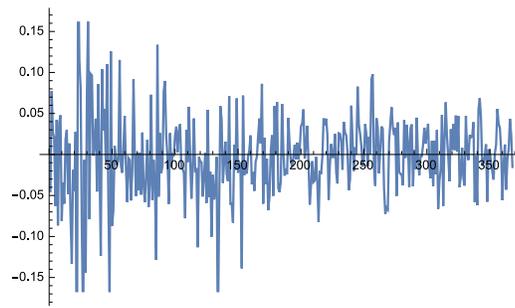
TABLE 1
The list of financial institutions.

i	Stock Code	Financial institution
1	8306	MUFG
2	8308	Resona Group (RG)
3	8316	Sumitomo Mitsui Financial Group (SMFG)
4	8411	Mizuho Financial Group (MFG)
5	8601	Daiwa Securities Group Inc. (DSGI)
6	8725	MS&AD Holdings (MH)

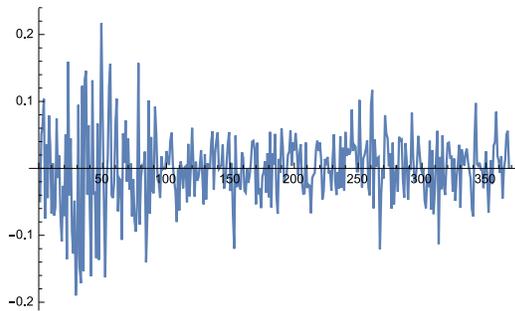
The observations are taken from the weekly closing price of all above institutions ranging from April 5, 2008 to May 2, 2015. The sample size is 370. The log return of each institution and the Nikkei index is used as the asset return. The log return of each institution $i = 1, \dots, 6$ is given in Figure 6 and that of the Nikkei index is given in Figure 7.



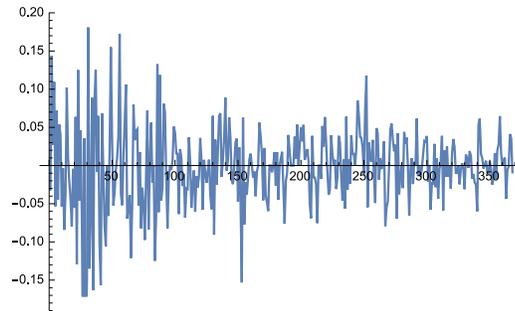
(a) Log return of MUFG.



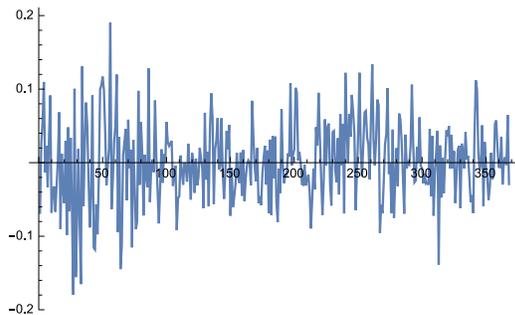
(b) Log return of RG.



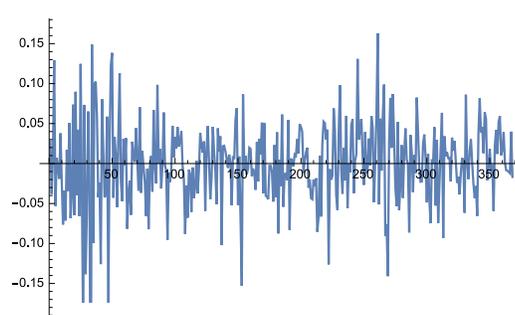
(c) Log return of SMFG.



(d) Log return of MFG.



(e) Log return of DSGI.



(f) Log return of MH.

Fig 6: Log returns of institutions $i = 1, \dots, 6$.

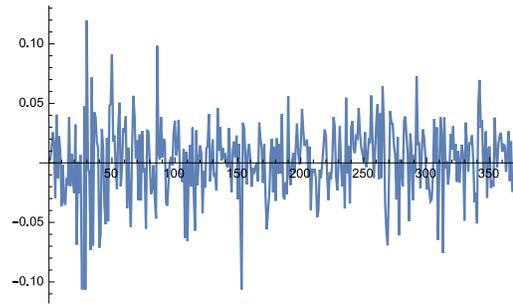
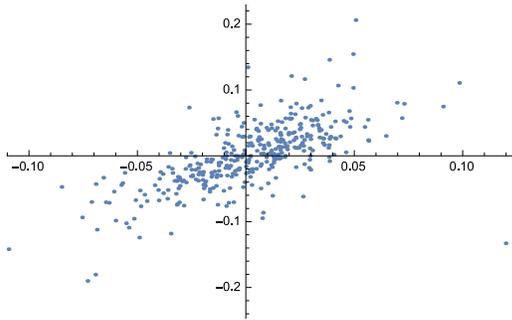
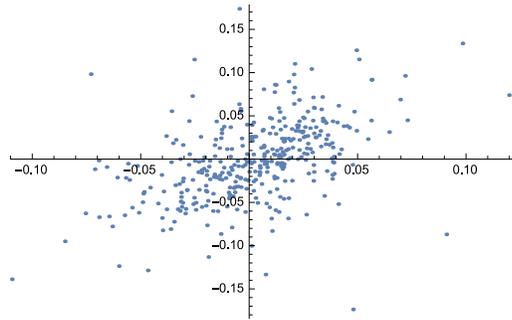


Fig 7: Log return of the Nikkei index.

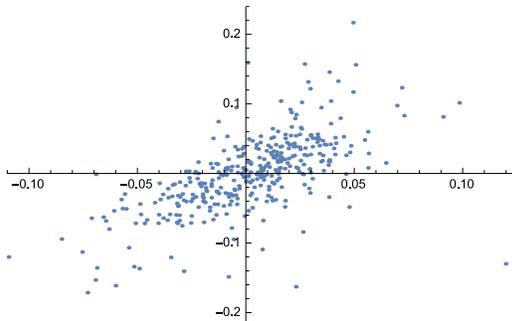
Next, we made the scatter plots between each institution and the Nikkei index in Figure 8 to see the interrelationship between the return of certain financial firm and that of market.



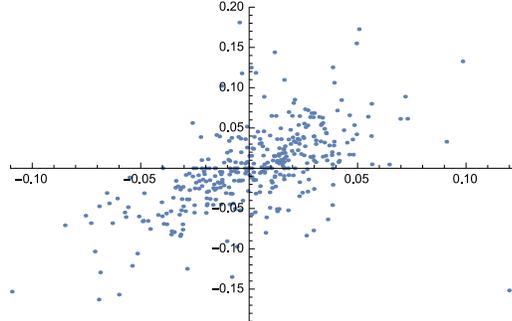
(a) Scatter plot of returns of Nikkei and MUFG.



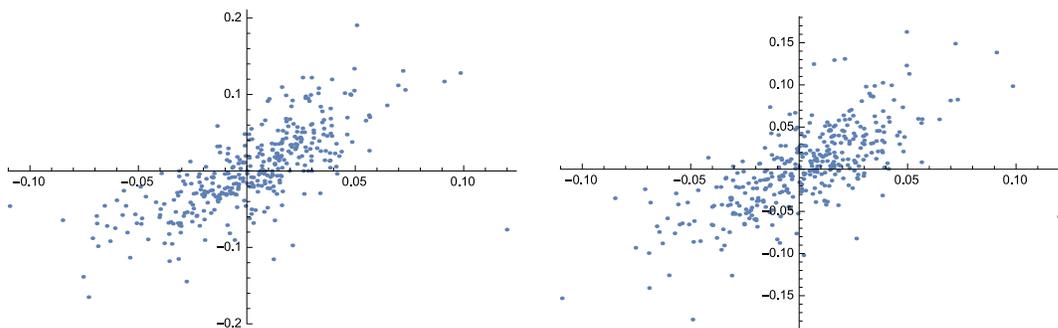
(b) Scatter plot of returns of Nikkei and RG.



(c) Scatter plot of returns of Nikkei and SMFG.



(d) Scatter plot of returns of Nikkei and MFG.



(e) Scatter plot of returns of Nikkei and DSGI.

(f) Scatter plot of returns of Nikkei and MH.

Fig 8: Scatter plots between return of Nikkei and that of institutions $i = 1, \dots, 6$.

Melchiori (2003) suggests the Gumbel-Hougaard copula for financial modeling. We summarized the estimated parameter d , in Gumbel-hougaard copula for each joint distribution between the log returns of i th institution and the Nikkei index, 5% VaR and 5%–1%, 5%–5% and 5%–10% CoVaR of the Nikkei index against each institution in Table 2. Here, each log return of institutions and the Nikkei index is supposed to be normal distribution.

TABLE 2
The Nikkei's VaR and CoVaR according to the Gumbel-Hougaard copula.

Institution	d	5% VaR	CoVaR(5%, 1%)	CoVaR(5%, 5%)	CoVaR(5%, 10%)
1	2.2121	-0.0552	-0.0961	-0.0818	-0.0737
2	1.6888	-0.0552	-0.0870	-0.0771	-0.0715
3	2.0687	-0.0552	-0.0937	-0.0806	-0.0731
4	1.9966	-0.0552	-0.0932	-0.0804	-0.0732
5	2.3479	-0.0552	-0.0945	-0.0798	-0.0716
6	2.3116	-0.0552	-0.0966	-0.0817	-0.0734

Next, we use new asymmetric copula (3.1) to model the joint returns of the Nikkei index and each institution i . As what we did above, we summarized the estimated parameter d in the new asymmetric copula for each joint distribution. In addition, we list 5% VaR and 5%–1%, 5%–5% and 5%–10% CoVaR of the Nikkei index against each institution in Table 3. Here, each log return of institutions and the Nikkei index is supposed to be normal distribution.

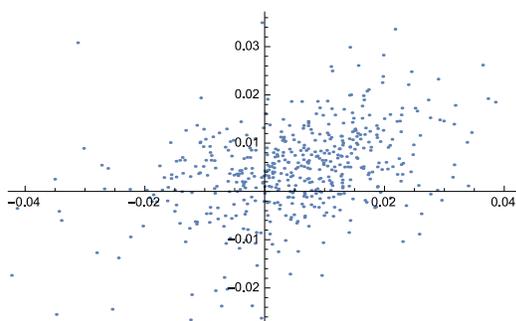
TABLE 3
The Nikkei's VaR and CoVaR according to new copula.

Institution	d	5% VaR	CoVaR(5%, 1%)	CoVaR(5%, 5%)	CoVaR(5%, 10%)
1	-1.0000	-0.0552	-0.0587	-0.0583	-0.0579
2	-1.0000	-0.0552	-0.0589	-0.0585	-0.0581
3	-1.0000	-0.0552	-0.0588	-0.0585	-0.0580
4	-1.0000	-0.0552	-0.0588	-0.0585	-0.0581
5	-1.0000	-0.0552	-0.0586	-0.0583	-0.0579
6	0.9969	-0.0552	-0.0867	-0.0658	-0.0563

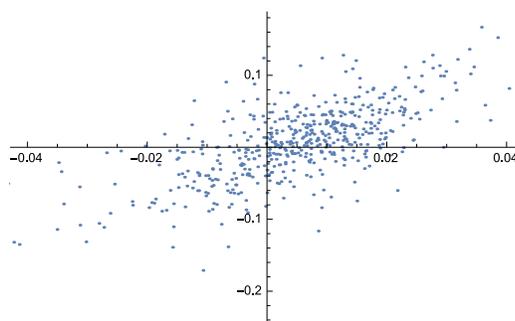
The estimates of parameter d have the same value of -1 for joint distribution of the Nikkei index and institution i ($i = 1, 2, 3, 4, 5$). Although the sixth institution has a quite different value of d , the CoVaR at any τ^* level has similar value to each other CoVaR. From Tables 2 and 3, we can see that the values of CoVaR are very stable for different financial institutions. This leads us to consider modeling the systemic risk by the measure CoVaR. In addition, from the property of the Nikkei index, all six institutions have the same weights in the Nikkei index. Thus, the new copula (3.1), which has more similar CoVaR, is more natural as a tool to evaluate CoVaR.

5. Optimal portfolio for the GPIF. In this section, we give our consideration to the optimal portfolio for the GPIF. Assets of the GPIF are composed of Domestic Bond (DB), Domestic Equity (DE), Foreign Bond (FB), Foreign Equity (FE) and Cash (C). The data of log returns are from December 31, 1970 to December 31, 2013. The sample size is 516.

First, we show the joint distribution of the GPIF and each asset by scatter plots in Figure 9. We suppose the portfolio of the GPIF is (60%, 12%, 11%, 12%, 5%) as what it was before the change of portfolio.



(a) Scatter plot of returns of the GPIF and DB.



(b) Scatter plot of returns of the GPIF and DE.

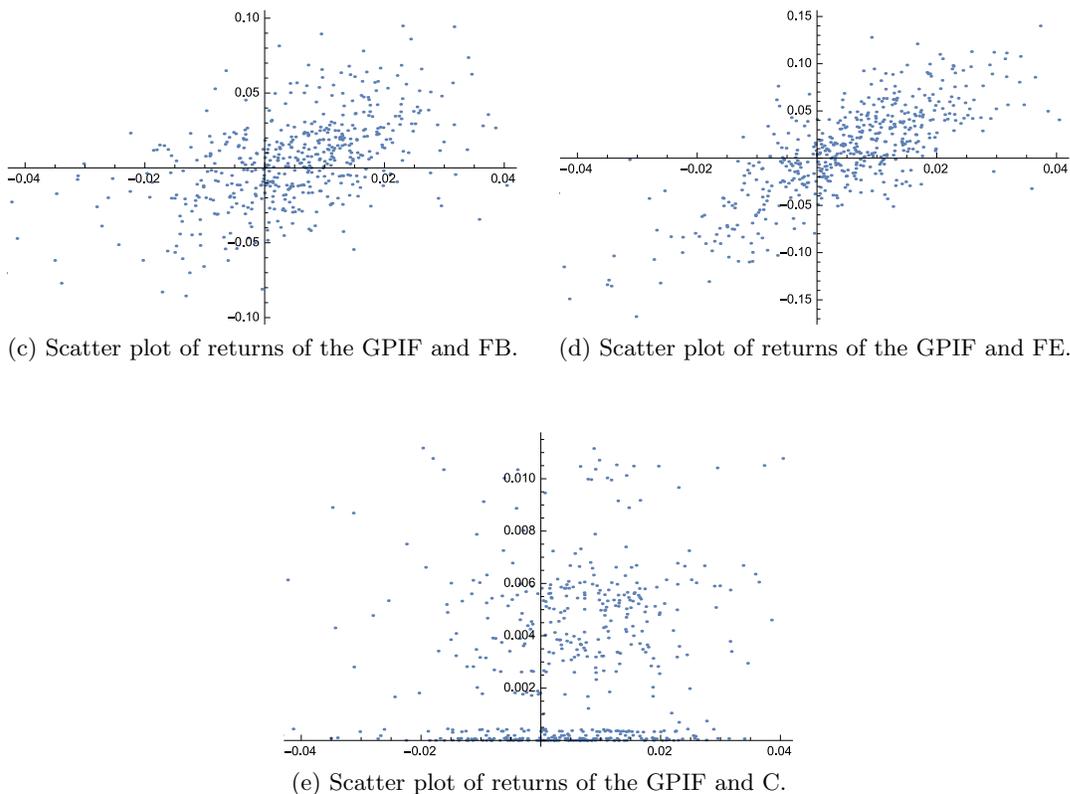


Fig 9: Scatter plots between return of the GPIF and each asset.

Interestingly, it seems that applying other distributions to the marginal distributions in copula function is more appropriate than normal distributions, although we found that the normal marginals are sufficient to model the log returns of financial institution's stocks. Let $t(\nu, \mu, \sigma)$ denote Student's t-distribution with ν degrees of freedom, shift μ and scale σ , and $U(a, b)$ denote uniform distribution whose support is $[a, b]$.

We used Gumbel-hougaard copula to model the joint distribution. In addition to parameter d in Gumbel-hougaard copula, we also report the estimated marginal distributions of each joint distribution, 5% VaR and 5%-1%, 5%-5% and 5%-10% CoVaR of the GPIF against each asset in Table 4. We omit CoVaR from 5%-1% CoVaR etc. in Tables 4 and 5 to save the space.

TABLE 4
the GPIF's VaR and CoVaR according to the Gumbel-Hougaard copula.

Asset	d	GPIF	Asset	5% VaR	5%-1%	5%-5%	5%-10%
DB	1.425	t(5.857, 0.005, 0.011)	t(3.518, 0.005, 0.010)	-0.0181	-0.0294	-0.0251	-0.0228
DE	1.984	t(6.968, 0.005, 0.011)	t(6.126, 0.007, 0.045)	-0.0181	-0.0376	-0.0292	-0.0251
FB	1.592	t(5.215, 0.005, 0.011)	t(4.486, 0.005, 0.025)	-0.0181	-0.0343	-0.0279	-0.0246
FE	2.270	t(5.983, 0.005, 0.012)	t(5.939, 0.008, 0.045)	-0.0181	-0.0431	-0.0318	-0.0265
C	1.119	t(5.190, 0.007, 0.011)	U(0.000, 0.011)	-0.0181	-0.0195	-0.0183	-0.0176

Next, we use new asymmetric copula (3.1) to model the joint returns of the GPIF and each asset. We report the estimated parameter d in Gumbel-hougaard copula, marginal distributions of each joint distribution, 5% VaR and 5%-1%, 5%-5% and 5%-10% CoVaR of the GPIF against each asset in Table 5.

TABLE 5
the GPIF's VaR and CoVaR according to new copula.

Asset	d	GPIF	Asset	5% VaR	5%-1%	5%-5%	5%-10%
DB	0.860	t(5.778, 0.005, 0.012)	t(4.199, 0.005, 0.010)	-0.0181	-0.0266	-0.0234	-0.0206
DE	-1.000	t(5.742, 0.005, 0.011)	t(6.437, 0.009, 0.044)	-0.0181	-0.0183	-0.0181	-0.0179
FB	-1.000	t(5.717, 0.005, 0.011)	t(5.325, 0.005, 0.025)	-0.0181	-0.0183	-0.0181	-0.0179
FE	-1.000	t(5.967, 0.005, 0.011)	t(5.490, 0.012, 0.042)	-0.0181	-0.0185	-0.0183	-0.0181
C	0.215	t(0.915, 0.005, 0.010)	U(-0.013, 0.011)	-0.0181	-0.0709	-0.0705	-0.0701

Suppose the log returns of DB, DE, FB and FE are denoted by X_1 , X_2 , X_3 and X_4 , respectively. Let the superscript T denote the transpose of the corresponding vector in the following. According to the new basic portfolio asset allocation of the GPIF, we define the optimal portfolio ω_{opt} for the weights $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T \in [0, 1]^4$ of X_1 , X_2 , X_3 and X_4 as follows:

$$\omega_{\text{opt}} = \arg \min_{\omega \in [0, 1]^4} -\omega^T \text{CoVaR}(\tau, \tau^*),$$

where $\text{CoVaR}(\tau, \tau^*) = (\text{CoVaR}_1(\tau, \tau^*), \text{CoVaR}_2(\tau, \tau^*), \text{CoVaR}_3(\tau, \tau^*), \text{CoVaR}_4(\tau, \tau^*))^T$ and $\sum_{i=1}^4 \omega_i = 1$. This optimization problem is well-defined but not trivial since $\text{CoVaR}(\tau, \tau^*)$ is not a linear or simple function of ω .

Let us first consider the risk measure $-\omega^T \text{CoVaR}(\tau, \tau^*)$ for the new basic portfolio (35%, 25%, 15%, 25%) of the GPIF. Under this portfolio, we summarized the estimates of parameter d in copula and the marginal distributions of each joint distribution in Table 6.

TABLE 6
Copula parameter and marginal distributions of the new basic portfolio.

Asset	d	GPIF	Asset
DB	0.000	t(5.142, 0.007, 0.020)	t(10.089, 0.005, 0.010)
DE	1.000	t(6.623, 0.007, 0.020)	t(7.793, 0.012, 0.043)
FB	0.890	t(5.088, 0.007, 0.020)	t(5.333, 0.006, 0.025)
FE	1.000	t(7.171, 0.008, 0.018)	t(7.408, 0.017, 0.040)

Next, we summarized the average conditional quantile $\omega^T \text{CoVaR}(\tau, \tau^*)$ of the new basic portfolio at different quantile levels (τ, τ^*) in Table 7.

TABLE 7
Average conditional quantile $\omega^T \text{CoVaR}(\tau, \tau^)$ ($\times 10^{-2}$) at different quantile levels (τ, τ^*).*

$\tau^* \backslash \tau$	1%	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
1%	-7.206	-5.023	-4.064	-3.450	-2.962	-2.527	-2.110	-1.677	-1.186	-0.536	0.734
5%	-5.858	-3.805	-2.901	-2.321	-1.859	-1.450	-1.060	-0.666	-0.246	0.221	0.734
10%	-5.304	-3.294	-2.414	-1.852	-1.408	-1.021	-0.662	-0.315	0.031	0.381	0.734
15%	-5.025	-3.024	-2.157	-1.608	-1.181	-0.813	-0.480	-0.165	0.139	0.438	0.734
20%	-4.876	-2.864	-2.003	-1.464	-1.049	-0.696	-0.380	-0.087	0.194	0.466	0.734
25%	-4.807	-2.768	-1.908	-1.375	-0.968	-0.626	-0.321	-0.041	0.225	0.482	0.734
30%	-4.789	-2.714	-1.851	-1.320	-0.918	-0.582	-0.285	-0.013	0.244	0.491	0.734
35%	-4.803	-2.689	-1.819	-1.288	-0.888	-0.556	-0.263	0.004	0.255	0.497	0.734
40%	-4.838	-2.684	-1.804	-1.271	-0.872	-0.541	-0.251	0.014	0.262	0.500	0.734
45%	-4.885	-2.694	-1.803	-1.266	-0.865	-0.534	-0.245	0.019	0.265	0.502	0.734
50%	-4.938	-2.714	-1.811	-1.269	-0.865	-0.533	-0.243	0.020	0.266	0.503	0.734

The tail probability pair (τ, τ^*) for a loss of 5.59% in asset return is plotted in Figure 10. 5.59% loss in asset return of the GPIF has happened in the second quarter in 2015.

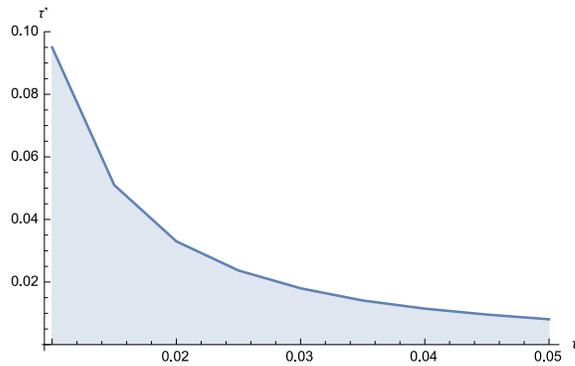


Fig 10: Tail probability pair (τ, τ^*) for a loss of 5.59% in asset return.

Finally, we used the new copula (3.1) to find the optimal portfolio of the GPIF. As preparation to avoid heavy computation, we directly estimated parameters μ and σ from each marginal samples since their estimates are quite accurate even if we do not estimate them simultaneously with other parameters. After estimating parameters d and ν for each portfolio, we obtained the following optimal portfolio ω_{opt} from the point of view of CoVaR(τ , 0.05) at different quantile level τ in Table 8.

TABLE 8
the optimal portfolio weights and the corresponding risk at different τ^* .

τ^*	τ	ω_1	ω_2	ω_3	ω_4	Risk
0.05	0.01	95.31%	1.35%	1.96%	1.37%	0.0256
0.05	0.05	95.00%	0.00%	3.01%	1.98%	0.0160
0.05	0.10	92.39%	0.59%	4.77%	2.24%	0.0115
0.05	0.15	90.52%	2.53%	3.59%	3.36%	0.0088
0.05	0.20	84.10%	3.78%	7.36%	4.75%	0.0072
0.05	0.25	87.00%	7.54%	2.64%	2.82%	0.0055
0.05	0.30	88.45%	4.23%	7.13%	0.18%	0.0029
0.05	0.35	89.68%	3.17%	3.27%	3.88%	0.0013
0.05	0.40	86.38%	5.95%	4.54%	3.14%	-0.0006
0.05	0.45	86.21%	3.74%	9.07%	0.98%	-0.0029
0.05	0.49	14.52%	24.24%	0.05%	61.19%	-0.0065

Comparing with Table 7, we find that the optimal portfolio of the GPIF improves the risk more at lower quantile than that at higher quantile. We have to put more weight on Domestic Bond if we fear a big failure in the GPIF although the way to put weights is not so trivial. We can analyze the optimal portfolio in more detail if we have more data and more concrete perspective on the risk management of the GPIF.

Acknowledgements. The author would like to thank Professor Masanobu Taniguchi for his advice to pay attention on R. Engle's work. The author also would like to thank Professor Xiaofeng Shao for his helpful introduction of the work in this direction, as well as a referee for his carefully reading the original manuscript.

References.

- ACHARYA, V., ENGLE, R. and RICHARDSON, M. (2012). Capital shortfall: A new approach to ranking and regulating systemic risks. *The American Economic Review* **102** 59–64.
- ADRIAN, T. and BRUNNERMEIER, M. K. (2011). CoVaR Technical Report, National Bureau of Economic Research.
- BERNARD, C. and CZADO, C. (2015). Conditional quantiles and tail dependence. *Journal of Multivariate Analysis* **138** 104–126.
- BOUYÉ, E. and SALMON, M. (2009). Dynamic copula quantile regressions and tail area dynamic dependence in Forex markets. *The European Journal of Finance* **15** 721–750.
- HUA, L. and JOE, H. (2014). Strength of tail dependence based on conditional tail expectation. *Journal of Multivariate Analysis* **123** 143–159.
- KOENKER, R. (2005). *Quantile regression* 38. Cambridge university press.

- KOENKER, R. and BASSETT, J. G. (1978). Regression quantiles. *Econometrica: journal of the Econometric Society* **46** 33–50.
- MELCHIORI, M. R. (2003). Which Archimedean copula is the right one? *YieldCurve*, October.
- NELSEN, R. B. (2007). *An introduction to copulas*. Springer Science & Business Media.
- SHAO, X. and ZHANG, J. (2014). Martingale difference correlation and its use in high-dimensional variable screening. *Journal of the American Statistical Association* **109** 1302–1318.
- WU, S. (2014). Construction of asymmetric copulas and its application in two-dimensional reliability modelling. *European Journal of Operational Research* **238** 476–485.

YAN LIU
DEPARTMENT OF APPLIED MATHEMATICS
SCHOOL OF FUNDAMENTAL SCIENCE AND ENGINEERING
WASEDA UNIVERSITY
3-4-1, OKUBO, SHINJUKU-KU, TOKYO, 169-8555
JAPAN
E-MAIL: y.liu2@kurenai.waseda.jp