WAVELET TRANSFORMATION

YAN LIU

1. Reference

謝, 鈴木 (2002).
Wojtaszczyk (1997).

2. Introduction

Both the Fourier transformation and the wavelet transformation transform the function from time domain to frequency domain. The main idea is based on the theory of the basis for functions. The difference between two methods is that the Fourier transformation is based on the basis

$$\mathfrak{B} = \{\mathfrak{b}(t - nq_0)e^{imp_0t}; m, n \in \mathbb{Z}\},$$

while the wavelet transformation is based on the basis

$$\mathfrak{B} = \{|p_0|^{-m/2}\psi(p_0^{-m}t - nq_0); m, n \in \mathbb{Z}\}.$$

Here, $\psi(\cdot)$ is called *mother wavelet*.

3. Theory for basis

3.1. the basis for wavelets.

Definition 3.1 (MRA)**.** The closed subspace $\{V_j; j \in \mathbb{Z}\} \subset L^2(\mathbb{R})$ is called *multiresolution analysis (MRA)* if

(i) $V_j \subset V_{j+1}, j \in \mathbb{Z}$,
(ii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $(\cup_{j\in\mathbb{Z}} V_j)^c = L^2(\mathbb{R})$,
(iii) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,
(iv) there exists a function $\varphi(x) \in V_0$ such that $\{\varphi(x - k); k \in \mathbb{Z}\}$ is the orthornormal basis for V_0.

Here, $\varphi(\cdot)$ is called *scaling function.*

Note that $L^2(\mathbb{R})$ can be always represented by

$$L^2(\mathbb{R}) = V_J \oplus \sum_{s=J}^{\infty} \oplus W_s.$$

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Define

$$h_k = \sqrt{2}\int_{\mathbb{R}} \varphi(x)\overline{\varphi(2x-k)}dx.$$

The scaling function $\varphi(x)$ satisfies

$$\varphi(x) = \sqrt{2}\sum_k h_k \varphi(2x-k).$$

Mother wavelet $\psi(x)$ is defined by

$$\psi(x) = \sum_k (-1)^k \overline{h}_{1-k}\varphi(2x-k).$$

As a result, any function $f(x) \in L^2(\mathbb{R})$ has a representation

$$\begin{aligned} f(x) &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} b_{j,k}\psi_{j,k}(x) \\ &= \sum_{k=-\infty}^{\infty} \alpha_{J,k}\varphi_{J,k}(x) + \sum_{j=J}^{\infty}\sum_{k=-\infty}^{\infty} \beta_{j,k}\psi_{j,k}(x). \end{aligned}$$

Obviously,

$$\begin{aligned} b_{j,k} &= \langle\, f, \psi_{j,k}\,\rangle, \quad j,k \in \mathbb{Z}, \\ \alpha_{J,k} &= \langle\, f, \varphi_{J,k}\,\rangle, \quad k \in \mathbb{Z}, \\ \beta_{j,k} &= \langle\, f, \psi_{j,k}\,\rangle, \quad j \geq J, k \in \mathbb{Z}. \end{aligned}$$

3.2. **norm.** Define $\tau_h f(x) = f(x-h)$. The Besov space $B^s_{p,q}$ is defined as follows: for $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$,

(1) for $s \in (0,1)$,

$$\begin{aligned} \gamma_{s,p,q}(f) &= \Big\{\int_{\mathbb{R}} \Big(\frac{\|\tau_h f - f\|_{L^p}}{|h|^s}\Big)^q \frac{dh}{|h|}\Big\}^{1/q}, \\ \gamma_{s,p,\infty}(f) &= \sup_{h\in\mathbb{R}} \frac{\|\tau_h f - f\|_{L^p}}{|h|^s}, \end{aligned}$$

(2) for $s = 1$,

$$\begin{aligned} \gamma_{1,p,q}(f) &= \Big\{\int_{\mathbb{R}} \Big(\frac{\|\tau_h f + \tau_{-h} f - 2f\|_{L^p}}{|h|}\Big)^q \frac{dh}{|h|}\Big\}^{1/q}, \\ \gamma_{1,p,\infty}(f) &= \sup_{h\in\mathbb{R}} \frac{\|\tau_h f + \tau_{-h} f - 2f\|_{L^p}}{|h|^s}, \end{aligned}$$

then we say

$$f \in B^s_{p,q} \iff \gamma_{s,p,q}(f) < \infty, f \in L^p(\mathbb{R}).$$

The Besov norm for $f(x) \in L^2(\mathbb{R}) \cap B^s_{2,q}$ is well defined and

$$\|f\|^s_{2,q} = \Big(\sum_k |\alpha_{0,k}|^2\Big)^{1/2} + \Big[\sum_{l \geq 0}\Big(2^{ls}\Big(\sum_k |\beta_{l,k}|^2\Big)^{1/2}\Big)^q\Big]^{1/q}$$

4. Further reading

Kato and Masry (1999) for wavelet transform of fractional Brownian motion, Donoho and Johnstone (1994, 1995) and Donoho et al. (1995, 1997) for wavelet for statistics. Also see Japanese work like Kawasaki and Shibata (1995) and Shibata and Takagiwa (1997).

5. Idea

The main purpose is to transmit functions using some finite device. Suppose $f(x) \in L^2$. It is known that f can be represented by the basis in L^2.

$$f = \sum_n a_n f_n.$$

Precisely, the wavelet is defined as follows.

Definition 5.1. A wavelet is a function $\Psi(t) \in L^2(\mathbb{R})$ such that the family of functions

$$\Psi_{j,k} = 2^{j/2}\Psi(2^j t - k)$$

where j and k are arbitrary integers, is an orthonormal basis in the Hillbert space $L^2(\mathbb{R})$.

Definition 5.2. a dyadic dilation operator J_s defined on $\mathbb{R}$ is given by

$$J_s(f)(x) = f(2^s x).$$

It is generalized by tensoring at the level of wavelets corresponds to dilations

$$J_{s_1,\ldots,s_D}(f)(x_1,\ldots,x_d) = f(2^{s_1}x_1,\ldots,2^{s_d}x_d),$$

or tensoring at the level of the scaling function corresponds to dilations

$$J_s(f)(x_1,\ldots,x_d) = f(2^s x_1,\ldots,2^s x_d).$$

More generally,

$$J_A(f)(x_1,\ldots,x_d) = f(A(x_1,\ldots,x_d)).$$

Remark 5.3. The multivariable theory is much less developed and much more complicated than the one-variable theory.

6. multivariable theory

6.1. Notations.

1. $X_j \subset L^2(\mathbb{R})$	a closed subspace
2. $(f^j_s)_{s\in A}$	the orthonormal bases in subspaces X_j

6.2. Operators.

$$1.\ U_A f(x) = |\det A|^{1/2} f(Ax)$$

6.3. **Fundamental Results.**

Lemma 6.1. *$\bigotimes_{j=1}^{d} X_j$ is a closed subspace of $L^2(\mathbb{R}^d)$. Also, $\left(\bigotimes_{j=1}^{d} f_{s_j}^{j}\right)_{(s_1,\dots,s_d)\in A_1\times\cdots\times A_d}$ is an orthonormal basis in $\bigotimes_{j=1}^{d} X_j$. Further, $\bigotimes_{j=1}^{d} L^2(\mathbb{R}) = L^2(\mathbb{R}^d)$.*

Proposition 6.2 (Wojtaszczyk (1997), Proposition 5.1)**.** *Let $(\Psi_j)_{j=1}^{d}$ be wavelets on $\mathbb{R}$ and let*

$$\Psi(x_1,\dots,x_d) = \prod_{j=1}^{d} \Psi_j(x_j).$$

Then the system

$$2^{\frac{j_1+\cdots+j_d}{2}} \Psi(2^{j_1}x_1 - k_1, \dots, 2^{j_d}x_d - k_d)$$

for all $j_1,\dots,j_d$ and $k_1,\dots,k_d$ in $\mathbb{Z}$ forms an orthonormal basis in $L^2(\mathbb{R}^d)$.

Remark 6.3. The problem is that the decay of elements of the basis in different directions can be markedly different. The remedy for the problem is to tensor multi resolution analysis but not wavelets.

Proposition 6.4 (Wojtaszczyk (1997), Proposition 5.2)**.** *Suppose we have d multi resolution analyses in $L^2(\mathbb{R})$ with scaling functions $\Phi^{0,j}(x)$ and associated wavelets $\Phi^{1,j}(x)$ for $j = 1,\dots,d$. Let $E = \{0,1\}^d \setminus (0,\dots,0)$. For $e = (e_1,\dots,e_d) \in E$ let $\Psi^e = \bigotimes_{j=1}^{d} \Phi^{e_j,j}$. Then the system*

$$\{2^{\frac{dj}{2}\Psi^e(2^j x-\gamma)}\}_{e\in E, j\in\mathbb{Z}, \gamma\in\mathbb{Z}^d}$$

is an orthonormal basis in $L^2(\mathbb{R})$.

Definition 6.5. A multiresolution analysis associated with a dilation matrix A is a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ satisfying

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$
(ii) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$
(iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$
(iv) $f \in V_j \iff f(Ax) \in V_{j+1}$, i.e., $V_j = U_A^j V_0$
(v) $f \in V_0 \iff f(x-\gamma) \in V_0$ for all $\gamma \in \mathbb{Z}^d$
(vi) there exists a function $\Phi \in V_0$ called a scaling function such that the system $\{\Phi(t-\gamma)_{\gamma\in\mathbb{Z}^d}\}$ is an orthonormal basis in V_0.

Proposition 6.6 (Wojtaszczyk (1997), Proposition 5.18)**.** *Let Q be a measurable subset of $\mathbb{R}^d$. Suppose that the function $c\mathbb{1}_Q$ is a scaling function of a multi resolution analysis associated with a dilation A. Then*

(i) *Q and $(Q+\gamma)$ are non-overlapping for all $\gamma \in \mathbb{Z}^d$, $\gamma \neq 0$.*
(ii) *there exists a set of digits, say $k_1,\dots,k_q$, such that $A(Q) = \cup_{i=1}^{q}(Q+k_i)$.*
(iii) *$\cup_{\gamma\in\mathbb{Z}^d}(Q+\gamma) = \mathbb{R}^d$.*

Conversely, if Q satisfies (i)-(iii) *then $\mathbb{1}_Q$ is a scaling function of multiresolution analysis associated with A.*

6.4. **Scaling function in $\mathbb{R}^d$.**

1. A	the dilation
2. $S = \{k_1, \dots, k_q\}$	a set of digits
3. $Q = \left\{ x \in \mathbb{R}^d \middle\| x = \sum_{j=1}^{\infty} A^{-j} s_j \text{ where } s_j \in S \right\}$	
4. $m(\xi) = \frac{1}{\|\det A\|} \sum_{k \in S} e^{-\langle \xi, k \rangle}$	

Proposition 6.7 (Wojtaszczyk (1997), Proposition 5.19)**.** *Under the notations above, we have*

(i) *Q is a compact subset of $\mathbb{R}^d$,*
(ii) $A(Q) = \cup_{i=1}^{q} (Q + k_i)$*,*
(iii) $\cup_{\gamma \in \mathbb{Z}^d} (Q + \gamma) = \mathbb{R}^d$*,*
(iv) *Q contains an open set.*

Corollary 6.8 (Wojtaszczyk (1997), Proposition 5.20)**.** The following statements are equivalent:

(i) $\mathbb{1}_Q$ is a scaling function of a multi resolution analysis
(ii) $|Q| = 1$
(iii) $|Q \cup (Q + \gamma)| = 0$ for every $\gamma \in \mathbb{Z}^d$, $\gamma \neq 0$.

A sufficient condition for $\mathbb{1}_Q$ to be a scaling function is given in the following proposition.

Proposition 6.9 (Wojtaszczyk (1997), Proposition 5.21)**.** *$\mathbb{1}_Q$ is a scaling function of multiresolution analysis if there exists a compact set $K \subset \mathbb{R}^d$ such that*

(i) *K contains a neighborhood of 0,*
(ii) $\cup_{\gamma \in \mathbb{Z}^d} (K + 2\pi\gamma) = \mathbb{R}^d$*,*
(iii) $|K \cap (K + 2\pi\gamma)| = 0$ *for all* $\gamma \in \mathbb{Z}^d$*,* $\gamma \neq 0$*,*
(iv) $m((A^*)^{-j}\xi) \neq 0$ *for all $\xi \in K$ and all integers $j > 0$.*

7. WORDS

1. archetypal	原型の、典型的な
2. recipient	受取人
3. acoustics	音響学
4. seismology	地震学
5. depict	表現する
6. for the time being	差し当たって
7. spring to mind	頭に浮かぶ
8. intrinsic	固有の
9. rudiments	基本原理
10. dyadic	一対の

11. dilation 拡張
12. at the outset はじめは

7.1. **phrases.**

(1) This should not prevent more mathematically experienced readers from starting their reading from this chapter if they wish to do so.