RANDOM FIELDS

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1. Reference

Tran (1990) JMA.
Hallin, Lu and Tran (2004) AS.

2. notations

2.1. Random Fields.

1. $\mathbb{Z}^N$	the N-dimensional integer lattice
2. $I_n = \{\boldsymbol{i} : \boldsymbol{i} \in \mathbb{Z}^N, 1 \le i_k \le n_k, k = 1, \ldots, N\}$	a rectangular region on $\mathbb{Z}^N$, $\boldsymbol{i}$ a site
3. $X_{\boldsymbol{j}} \in \mathbb{R}^d$, where $\boldsymbol{j} \in I_n$	a random field indexed by $\mathbb{Z}^N$
4. $f(x)$	the density function of X_n
5. $\hat{\boldsymbol{n}} = n_1 \cdots n_N$	size of random fields
6. $\hat{d}(S, S') := \min\{\|\boldsymbol{i} - \boldsymbol{i}'\|; \boldsymbol{i} \in S, \boldsymbol{i}' \in S'\}$	the Euclidean distance between S and S'
7. $\hat{f} : \mathbb{N}^2 \to \mathbb{R}^+$	a symmetric function nondecreasing in each variable

2.2. Kernel Density Estimator.

(i) the kernel density estimator

$$f_n(x) = (\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{-1} \sum_{\substack{j_k=1 \\ k=1,\ldots,N}}^{n_k} K\Big(\frac{x - X_{\boldsymbol{j}}}{b_{\boldsymbol{n}}}\Big)$$

(ii) Let

$$\sigma^2 = f(x) \int_{\mathbb{R}^d} K^2(u)du.$$

2.3. Fundamental Conditions.

(a1) Nonisotropic divergence ($\boldsymbol{n} \to \infty$)

$$\boldsymbol{n} \to \infty \quad \text{if } \min\{n_k\} \to \infty.$$

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(a2) Isotropic divergence ($\boldsymbol{n} \Rightarrow \infty$)

$$\left|\frac{n_j}{n_k}\right| \leq C \quad \text{for some } 0 < C < \infty,\ 1 \leq j, k \leq N.$$

3. Asymptotic normality of $f_n(x)$

3.1. Assumptions.

(A1) $\{X_{\boldsymbol{j}} \in \mathbb{R}^d; \boldsymbol{i} \in \mathbb{Z}^N\}$ is strictly stationary.
(A2) The joint probability density $f_{\boldsymbol{i},\boldsymbol{j}}(x, y)$ of $X_{\boldsymbol{i}}$ and $X_{\boldsymbol{j}}$ exists and satisfies $|f_{\boldsymbol{i},\boldsymbol{j}}(x, y) - f(x)f(y)| \leq C$ for some constant C and for all x, y and $\boldsymbol{i}$, $\boldsymbol{j}$.

(B1) $|K(x)|$ is uniformly bounded by a constant $\tilde{K}$.
(B2) Assume K has an integrable radial majorant, that is, $Q(x) \equiv \sup\{K(y) : \|y\| > \|x\|\}$ is integrable.
(B3) (B3-1)

$$\int_{\mathbb{R}^d} |K(x)|dx < \infty.$$

(B3-2) (1)

$$\int_{\mathbb{R}^d} \|x\||K(x)|dx < \infty.$$

(2) Suppose K is a probability density function on $\mathbb{R}^d$ and for any $x, y \in \mathbb{R}^d$ and some constant $\rho > 0$

$$|f(x) - f(y)| \leq \rho\|x - y\|.$$

(C1) There exists a function $\varphi(t) \downarrow 0$ as $t \to \infty$, such that whenever S, $S' \subset \mathbb{Z}^N$,

$$\begin{aligned}\alpha(\mathcal{B}(S), \mathcal{B}(S')) &= \sup\{|P(AB) - P(A)P(B)|, A \in \mathcal{B}(S), \mathcal{B} \in \mathcal{B}(S')\} \\ &\leq \hat{f}(\mathrm{Card}(S), \mathrm{Card}(S'))\varphi(\hat{d}(S, S')).\end{aligned}$$

(C2) (C2-1)

$$\hat{f}(n, m) \leq \min\{m, n\}.$$

(C2-2) for some $\tilde{k} > 1$ and some $C > 0$,

$$\hat{f}(n, m) \leq C(n + m + 1)^{\tilde{k}}.$$

Remark 3.1. If $\tilde{f} \equiv 1$, then $\{X_n\}$ is called strongly mixing.

Remark 3.2. (A4) is an assumption to control the fastness of convergence of $Ef_n(x)$ to $f(x)$.

3.2. Asymptotic normality of $f_n(x)$.

Theorem 3.3 (Tran (1990), Theorem 3.1). *Let (a2) hold with (C1) and (C2-1) with*

$$\varphi(x) = O(x^{-\mu})$$

for some $\mu > 2N$. *Let* $0 < \gamma < (\mu - N)\mu^{-1}$. *If Assumptions (A1), (A2), (B1), (B2) and (B3-1) hold and there exists a sequence of positive integers* $q = q_{\boldsymbol{n}} \to \infty$ *such that*

$$\begin{aligned} &q = o((\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)}), \\ &\hat{\boldsymbol{n}} q^{-\mu} \to 0, \\ &b_{\boldsymbol{n}}^{-d(1-\gamma)} q^{N-\mu(1-\gamma)} \to 0, \end{aligned}$$

then

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big(\frac{f_{\boldsymbol{n}}(x) - E f_{\boldsymbol{n}}(x)}{\sigma} \Big)$$

has a standard normal distribution as $\boldsymbol{n} \to \infty$.

Theorem 3.4 (Tran (1990), Theorem 3.2). *Let (a2) hold with (C1) and (C2-1) with*

$$\varphi(x) = O(e^{-\xi x})$$

for some $\xi > 0$. *Let* $0 < \gamma < 1$. *If Assumptions (A1), (A2), (B1), (B2) and (B3-1) hold and*

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)} (\log \hat{\boldsymbol{n}})^{-1} \to \infty,$$

then

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big(\frac{f_{\boldsymbol{n}}(x) - E f_{\boldsymbol{n}}(x)}{\sigma} \Big)$$

has a standard normal distribution as $\boldsymbol{n} \to \infty$.

Theorem 3.5 (Tran (1990), Theorem 3.3). *Let (a2) hold with (C1) and (C2-1) with conditions on* $\varphi(x)$ *as in Theorem 3.3 and 3.4. If Assumptions (A1), (A2), (B1), (B2) and (B3-2) hold with*

$$\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^{d+2} \to 0,$$

then

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big(\frac{f_{\boldsymbol{n}}(x) - E f_{\boldsymbol{n}}(x)}{\sigma} \Big)$$

has a standard normal distribution as $\boldsymbol{n} \to \infty$.

Theorem 3.6 (Tran (1990), Theorem 4.1). *Let (a2) hold with (C1) and (C2-2) with*

$$\varphi(x) = O(x^{-\mu})$$

for some $\mu > 2N$. *Let* $0 < \gamma < (\mu - N)\mu^{-1}$. *If Assumptions (A1), (A2), (B1), (B2) and (B3-1) hold and there exists a sequence of positive integers* $q = q_{\boldsymbol{n}} \to \infty$ *such that*

$$\begin{aligned} &q = o((\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)}), \\ &\hat{\boldsymbol{n}} (\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{(\tilde{k}-1)/2} q^{-\mu} \to 0, \\ &b_{\boldsymbol{n}}^{-d(1-\gamma)} q^{N-\mu(1-\gamma)} \to 0, \end{aligned}$$

then

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2}\Big(\frac{f_{\boldsymbol{n}}(x) - Ef_{\boldsymbol{n}}(x)}{\sigma}\Big)$$

has a standard normal distribution as $\boldsymbol{n} \to \infty$.

4. LOCAL LINEAR SPATIAL REGRESSION

4.1. Notations.

1. $g : x \mapsto g(x) := E[\,Y_{\boldsymbol{i}} | \boldsymbol{X_i} = \boldsymbol{x}\,]$ the spatial regression function $g : \mathbb{R}^d \to \mathbb{R}$.

4.2. A weighted least square estimator.

(1) A weighted least square estimator

$$\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) \\ g'_{\boldsymbol{n}}(\boldsymbol{x}) \end{pmatrix} = \arg \min_{(a_0, \boldsymbol{a}_1) \in \mathbb{R}^{d+1}} \sum_{\substack{j_k=1 \\ k=1,\dots,N}}^{n_k} (Y_{\boldsymbol{j}} - a_0 - \boldsymbol{a}_1^T(\boldsymbol{X_j} - \boldsymbol{x}))^2 K\Big(\frac{\boldsymbol{X_j} - \boldsymbol{x}}{b_{\boldsymbol{n}}}\Big)$$

(2) Let

$$\boldsymbol{U} = f(\boldsymbol{x}) \begin{pmatrix} \int_{\mathbb{R}^d} K(\boldsymbol{u}) d\boldsymbol{u} & \int_{\mathbb{R}^d} \boldsymbol{u}^T K(\boldsymbol{u}) d\boldsymbol{u} \\ \int_{\mathbb{R}^d} \boldsymbol{u} K(\boldsymbol{u}) d\boldsymbol{u} & \int_{\mathbb{R}^d} \boldsymbol{u}\boldsymbol{u}^T K(\boldsymbol{u}) d\boldsymbol{u} \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \mathrm{Var}(Y_{\boldsymbol{j}} | \boldsymbol{X_j} = \boldsymbol{x}) f(\boldsymbol{x}) \begin{pmatrix} \int_{\mathbb{R}^d} K^2(\boldsymbol{u}) d\boldsymbol{u} & \int_{\mathbb{R}^d} \boldsymbol{u}^T K^2(\boldsymbol{u}) d\boldsymbol{u} \\ \int_{\mathbb{R}^d} \boldsymbol{u} K^2(\boldsymbol{u}) d\boldsymbol{u} & \int_{\mathbb{R}^d} \boldsymbol{u}\boldsymbol{u}^T K^2(\boldsymbol{u}) d\boldsymbol{u} \end{pmatrix}.$$

For

$$\begin{aligned} g_{ij}(\boldsymbol{x}) &= \frac{\partial^2 g(\boldsymbol{x})}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, d, \\ \boldsymbol{u} &= (u_1, \cdots, u_d)^T, \end{aligned}$$

$$\begin{aligned} B_0(\boldsymbol{x}) &= \frac{1}{2} f(\boldsymbol{x}) \sum_{i=1}^d \sum_{j=1}^d g_{ij}(\boldsymbol{x}) \int u_i u_j K(\boldsymbol{u}) d\boldsymbol{u}, \\ B_1(\boldsymbol{x}) &= \frac{1}{2} f(\boldsymbol{x}) \sum_{i=1}^d \sum_{j=1}^d g_{ij}(\boldsymbol{x}) \int u_i u_j \boldsymbol{u} K(\boldsymbol{u}) d\boldsymbol{u}. \end{aligned}$$

(3) Further, define

$$\begin{aligned} B_g(\boldsymbol{x}) &= \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d g_{ii}(\boldsymbol{x}) \int u_i^2 K(\boldsymbol{u}) d\boldsymbol{u}, \\ \sigma_0^2(\boldsymbol{x}) &= \mathrm{Var}(Y_{\boldsymbol{j}} | \boldsymbol{X_j} = \boldsymbol{x}) (f(\boldsymbol{x}))^{-1} \int_{\mathbb{R}^d} K^2(\boldsymbol{u}) d\boldsymbol{u}, \\ \sigma_1^2(\boldsymbol{x}) &= \mathrm{Var}(Y_{\boldsymbol{j}} | \boldsymbol{X_j} = \boldsymbol{x}) (f(\boldsymbol{x}))^{-1} \Big(\int_{\mathbb{R}^d} \boldsymbol{u}\boldsymbol{u}^T K(\boldsymbol{u}) d\boldsymbol{u}\Big)^{-1} \Big(\boldsymbol{u}\boldsymbol{u}^T K^2(\boldsymbol{u}) d\boldsymbol{u}\Big) \Big(\int_{\mathbb{R}^d} \boldsymbol{u}\boldsymbol{u}^T K(\boldsymbol{u}) d\boldsymbol{u}\Big)^{-1}. \end{aligned}$$

4.3. **Modified Kernel.** For any $\boldsymbol{c} := (c_0, \boldsymbol{c}_1^T)^T \in \mathbb{R}^{d+1}$, define

$$K_{\boldsymbol{c}}(\boldsymbol{u}) := (c_0 + \boldsymbol{c}_1^T \boldsymbol{u}) K(\boldsymbol{u}).$$

4.4. **Assumptions.**

(A1) Replace $X_{\boldsymbol{j}} \in \mathbb{R}^d$ by $(Y_{\boldsymbol{j}}, X_{\boldsymbol{j}}) \in \mathbb{R}^{d+1}$ in (A1).
(A2) (A2)
(A3) Y_i has finite absolute moment of order $2+\delta$, that is, $E|Y_i|^{2+\delta} < \infty$ for some $\delta > 0$.
(A4) g is twice differentiable and g'' is continuous at all x.
(A5) (A5-1) (C2-1)
(A5-2) (C2-2)

(B1) Replace $K(x)$ by $K_{\boldsymbol{c}}(\boldsymbol{x})$ in (B1).
(B2) Replace $K(x)$ by $K_{\boldsymbol{c}}(\boldsymbol{x})$ in (B3-1).
(B3) For any $\boldsymbol{c} \in \mathbb{R}^{d+1}$, $K_{\boldsymbol{c}}(\boldsymbol{x})$ has an integrable second-order radial majorant, that is,

$$Q_{\boldsymbol{c}}^K(\boldsymbol{x}) \equiv \sup_{\|\boldsymbol{y}\| > \|\boldsymbol{x}\|} \{\|y\|^2 K_{\boldsymbol{c}}(\boldsymbol{y})\}$$

4.5. **Asymptotic normality of local linear spatial regression.**

Theorem 4.1 (Hallin, Lu and Tran (2004), Theorem 3.1)**.** *Let (A5-1) hold with*

$$\varphi(x) = O(x^{-\mu})$$

for some $\mu > 2(3+\delta)N/\delta$. Let $(4+\delta)N/(2+\delta) < \gamma < \mu\delta/(2+\delta) - N$. If Assumptions (A1)-(A4) and (B1)-(B3) hold and there exists a sequence of positive integers $q = q_{\boldsymbol{n}} \to \infty$ such that as $\boldsymbol{n} \Rightarrow \infty$,

$$q = o((\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/(2N)}),$$
$$\hat{\boldsymbol{n}} q^{-\mu} \to 0,$$
$$q b_{\boldsymbol{n}}^{\delta d/(\gamma(2+\delta))} > 1,$$

then as $\boldsymbol{n} \Rightarrow \infty$

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big[\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} - \boldsymbol{U}^{-1} \begin{pmatrix} B_0(\boldsymbol{x}) \\ B_1(\boldsymbol{x}) \end{pmatrix} b_{\boldsymbol{n}}^2 \Big] \xrightarrow{\mathcal{L}} \mathcal{N}(\boldsymbol{0}, \boldsymbol{U}^{-1} \boldsymbol{\Sigma} (\boldsymbol{U}^{-1})^T).$$

If the kernel $K(\cdot)$ is a symmetric density function, then

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) - B_g(\boldsymbol{x}) b_{\boldsymbol{n}}^2 \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\Big(\boldsymbol{0}, \begin{pmatrix} \sigma_0^2(\boldsymbol{x}) & 0 \\ 0 & \sigma_1^2(\boldsymbol{x}) \end{pmatrix}\Big).$$

Remark 4.2. Note that if the order of finite moments $\delta \to 0$, then

$$2(3+\delta)N/\delta \to 2N.$$

Theorem 4.3 (Hallin, Lu and Tran (2004), Theorem 3.2)**.** *Let (A5-1) hold with*

$$\varphi(x) = O(e^{-\xi x})$$

for some $\xi > 0$. *Let* $\gamma > (4+\delta)N/(2+\delta)$. *If Assumptions (A1)-(A4) and (B1)-(B3) hold and there exists a sequence of positive integers* $q = q_{\boldsymbol{n}} \to \infty$ *such that as* $\boldsymbol{n} \Rightarrow \infty$,

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^{d(1+2N\delta/(\gamma(2+\delta)))})^{1/(2N)} (\log \hat{\boldsymbol{n}})^{-1} \to \infty,$$

then as $\boldsymbol{n} \Rightarrow \infty$

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big[\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} - \boldsymbol{U}^{-1} \begin{pmatrix} B_0(\boldsymbol{x}) \\ B_1(\boldsymbol{x}) \end{pmatrix} b_{\boldsymbol{n}}^2 \Big] \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \boldsymbol{U}^{-1} \boldsymbol{\Sigma} (\boldsymbol{U}^{-1})^T).$$

If the kernel $K(\cdot)$ *is a symmetric density function, then*

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) - B_g(\boldsymbol{x}) b_{\boldsymbol{n}}^2 \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\Big(\mathbf{0}, \begin{pmatrix} \sigma_0^2(\boldsymbol{x}) & 0 \\ 0 & \sigma_1^2(\boldsymbol{x}) \end{pmatrix}\Big).$$

Theorem 4.4 (Hallin, Lu and Tran (2004), Theorem 3.3)**.** *Let (A5-1) hold with*

$$\varphi(x) = O(x^{-\mu})$$

for some $\mu > 2(3+\delta)N/\delta$. *Let* $(4+\delta)N/(2+\delta) < \gamma < \mu\delta/(2+\delta) - N$ *and* $b_{\boldsymbol{n}} = \prod_{i=1}^N b_{n_i}$. *If Assumptions (A1)-(A4) and (B1)-(B3) hold and there exists a sequence of positive integers* $q = q_{\boldsymbol{n}} \to \infty$ *such that as*

$$\begin{aligned} & q = o(\min_{1 \le k \le N} (n_k b_{n_k}^d)^{1/2}), \\ & \hat{\boldsymbol{n}} q^{-\mu} \to 0, \\ & q b_{\boldsymbol{n}}^{\delta d/(\gamma(2+\delta))} > 1, \end{aligned}$$

then as $\boldsymbol{n} \to \infty$

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big[\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} - \boldsymbol{U}^{-1} \begin{pmatrix} B_0(\boldsymbol{x}) \\ B_1(\boldsymbol{x}) \end{pmatrix} b_{\boldsymbol{n}}^2 \Big] \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \boldsymbol{U}^{-1} \boldsymbol{\Sigma} (\boldsymbol{U}^{-1})^T).$$

If the kernel $K(\cdot)$ *is a symmetric density function, then*

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) - B_g(\boldsymbol{x}) b_{\boldsymbol{n}}^2 \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\Big(\mathbf{0}, \begin{pmatrix} \sigma_0^2(\boldsymbol{x}) & 0 \\ 0 & \sigma_1^2(\boldsymbol{x}) \end{pmatrix}\Big).$$

Theorem 4.5 (Hallin, Lu and Tran (2004), Theorem 3.4)**.** *Let (A5-1) hold with*

$$\varphi(x) = O(e^{-\xi x})$$

for some $\xi > 0$. *Let* $\gamma > (4+\delta)N/(2+\delta)$ *and* $b_{\boldsymbol{n}} = \prod_{i=1}^N b_{n_i}$. *If Assumptions (A1)-(A4) and (B1)-(B3) hold and there exists a sequence of positive integers* $q = q_{\boldsymbol{n}} \to \infty$ *such that*

$$\min_{1 \le k \le N} \{(n_k b_{n_k}^d)^{1/2}\} b_{\boldsymbol{n}}^{d\delta/(\gamma(2+\delta))} (\log \hat{\boldsymbol{n}})^{-1} \to \infty,$$

then as $\boldsymbol{n} \to \infty$

$$(\hat{\boldsymbol{n}} b_{\boldsymbol{n}}^d)^{1/2} \Big[\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x}) - g(\boldsymbol{x}) \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x}) - g'(\boldsymbol{x})) \end{pmatrix} - \boldsymbol{U}^{-1} \begin{pmatrix} B_0(\boldsymbol{x}) \\ B_1(\boldsymbol{x}) \end{pmatrix} b_{\boldsymbol{n}}^2 \Big] \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \boldsymbol{U}^{-1} \boldsymbol{\Sigma} (\boldsymbol{U}^{-1})^T).$$

If the kernel $K(\cdot)$ *is a symmetric density function, then*

$$(\hat{\boldsymbol{n}}b_{\boldsymbol{n}}^d)^{1/2}\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x})-g(\boldsymbol{x})-B_g(\boldsymbol{x})b_{\boldsymbol{n}}^2 \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x})-g'(\boldsymbol{x}))\end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\Big(\boldsymbol{0}, \begin{pmatrix}\sigma_0^2(\boldsymbol{x}) & 0 \\ 0 & \sigma_1^2(\boldsymbol{x})\end{pmatrix}\Big).$$

Theorem 4.6 (Hallin, Lu and Tran (2004), Theorem 3.5)**.** *Let (A5-2) hold with*

$$\varphi(x) = O(x^{-\mu})$$

for some $\mu > 2(3+\delta)N/\delta$*. Let* $(4+\delta)N/(2+\delta) < \gamma < \mu\delta/(2+\delta) - N$*. If Assumptions (A1)-(A4) and (B1)-(B3) hold and there exists a sequence of positive integers* $q = q_{\boldsymbol{n}} \to \infty$ *such that as* $\boldsymbol{n} \Rightarrow \infty$*,*

$$\begin{aligned} &q = o((\hat{\boldsymbol{n}}b_{\boldsymbol{n}}^d)^{1/(2N)}),\\ &\hat{\boldsymbol{n}}^{\tilde{k}+1}q^{-\mu-N} \to 0,\\ &qb_{\boldsymbol{n}}^{\delta d/(\gamma(2+\delta))} > 1,\end{aligned}$$

then as $\boldsymbol{n} \Rightarrow \infty$

$$(\hat{\boldsymbol{n}}b_{\boldsymbol{n}}^d)^{1/2}\Big[\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x})-g(\boldsymbol{x}) \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x})-g'(\boldsymbol{x}))\end{pmatrix} - \boldsymbol{U}^{-1}\begin{pmatrix}B_0(\boldsymbol{x})\\ B_1(\boldsymbol{x})\end{pmatrix} b_{\boldsymbol{n}}^2\Big] \xrightarrow{\mathcal{L}} \mathcal{N}(\boldsymbol{0}, \boldsymbol{U}^{-1}\boldsymbol{\Sigma}(\boldsymbol{U}^{-1})^T).$$

If the kernel $K(\cdot)$ *is a symmetric density function, then*

$$(\hat{\boldsymbol{n}}b_{\boldsymbol{n}}^d)^{1/2}\begin{pmatrix} g_{\boldsymbol{n}}(\boldsymbol{x})-g(\boldsymbol{x})-B_g(\boldsymbol{x})b_{\boldsymbol{n}}^2 \\ b_{\boldsymbol{n}}(g'_{\boldsymbol{n}}(\boldsymbol{x})-g'(\boldsymbol{x}))\end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\Big(\boldsymbol{0}, \begin{pmatrix}\sigma_0^2(\boldsymbol{x}) & 0 \\ 0 & \sigma_1^2(\boldsymbol{x})\end{pmatrix}\Big).$$

5. words

6. Further Reading

6.1. nonparametric regression.