QR WITH LDP

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1. Reference

Koul and Mukherjee (1994), JMA

2. Notations

1.	$\boldsymbol{X}$	the $n \times p$ design matrix of known constants
2.	$\boldsymbol{x}_{ni}$	ith row of $\boldsymbol{X}$, $(1 \times p)$-matrix
3.	F	the common d.f. of $\{\epsilon_i\}$
4.	$\psi = (c, d) \subset \mathbb{R}$	$= \{x \in \mathbb{R}; 0 < F(x) < 1\}$
5.	G	a measurable function from $\mathbb{R}$ to $\mathbb{R}$
6.	D_c	$= (C'C)^{1/2}$, where C is of the full rank.
7.	$\\|\boldsymbol{X}_n\\|_a$	$= \sup\{\\|\boldsymbol{X}_n(\alpha)\\|,\ a \le \alpha \le 1 - a\}$
8.	$\boldsymbol{X}_n = O_P^*(1)$	$\equiv \\|\boldsymbol{X}_n\\|_a = O_P(1)$ for every $a \in (0, 1/2]$.

3. Concepts and definitions

3.1. model.

$\eta_1, \eta_2, \ldots$ stationary mean zero unit variance Gaussian process with

$$\rho(k) = E\eta_1\eta_{k+1}.$$

The response variable $\{Y_{ni}\}$ satisfying

$$Y_{ni} = \boldsymbol{x}_{ni}'\boldsymbol{\beta} + \epsilon_i, \quad 1 \le i \le n, \quad \boldsymbol{\beta} \in \mathbb{R}^p,$$

with $\epsilon_i = G(\eta_i)$.

3.2. long range dependence.

$$\rho(k) = k^{-\theta}L(k), \quad \text{for some } 0 < \theta < 1,\ \forall k \ge 1,$$

where $L(k)$ is positive for large k and is slowly varying at infinity.

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3.3. exponent. m is the Hermite rank of the class of functions.

$$J_q(x) = E\{\mathbb{1}(G(\eta) \geq x) - F(x)\}H_q(\eta).$$

Also, suppose $m(x) = \min\{q \geq 1; J_q \neq 0\}$, then

$$m = \inf\{m(x); x \in \psi\}.$$

Also, define $J_m^+(x)$ as

$$J_m^+(x) = E\mathbb{1}(G(\eta) \leq x)|H_m(\eta)|.$$

3.4. normalizing order. Assume $0 < \theta < 1/m$,

$$\tau_n = n^{(1-m\theta)/2}L^{m/2}(n), \quad n \geq 1.$$

3.5. regression transform.

$$\boldsymbol{\beta}(\alpha) = \boldsymbol{\beta} + F^{-1}(\alpha)\boldsymbol{e}_1, \quad \boldsymbol{e}_1 = (1, 0, \ldots, 0)',$$

and

$$q(\alpha) = f(F^{-1}(\alpha)), \quad 0 \leq \alpha \leq 1.$$

3.6. minimum-distance type estimator of $\boldsymbol{\beta}$.

$$\hat{\boldsymbol{\beta}}_{md}(\alpha) = \arg\min_t \|D_x^{-1}T(\boldsymbol{t}, \alpha)\|^2,$$

where

$$T(\boldsymbol{t}, \alpha) = \sum_i \boldsymbol{x}_{ni}\{\mathbb{1}(Y_{ni} - \boldsymbol{x}_{ni}'\boldsymbol{t} \leq 0) - \alpha\}, 0 \leq \alpha \leq 1, \quad \boldsymbol{t} \in \mathbb{R}^d.$$

3.7. αth regression quantile.

$$\hat{\boldsymbol{\beta}}_n(\alpha) = \hat{\boldsymbol{B}}(\alpha) = \left\{ \boldsymbol{b} \in \mathbb{R}^p : \sum_{i=1}^n h_\alpha(Y_{ni} - \boldsymbol{x}_{ni}'\boldsymbol{b}) = \text{minimum} \right\},$$

where $h_\alpha(u) = \alpha u \mathbb{1}(u > 0) - (1-\alpha)u\mathbb{1}(u \leq 0)$, $u \in \mathbb{R}$, $0 \leq \alpha \leq 1$.

4. Assumptions

4.1. Assumptions on γ_{ni} and ξ_{ni}.

(GX0-1) $\sum_{i=1}^n \gamma_{ni}^2 = 1, \forall n \geq 1$

(GX0-2) $\max_{1 \leq i \leq n} n^{1/2}|\gamma_{ni}| = O(1)$

(GX1-1) $\sum_{i=1}^n \gamma_{ni}^2 = 1, \forall n \geq 1$

(GX1-2) $\max_{1 \leq i \leq n} n^{1/2}|\gamma_{ni}| = O(1)$

(GX1-3) $\max_{1 \leq i \leq n}|\xi_i| = o(1)$

(GX2-1) F has a continuous density f.

(GX2-2) $\tau_n^{-1}\sum_{i=1}^n|\gamma_{ni}\xi_{ni}| = O(1)$.

(GX3-1) F has a continuous and positive density f.

(GX3-2) The functions $J_m(x)$ and $J_m^+(x)$ are continuously differentiable with derivatives equal to $\dot{J}_m(x)$ and $\dot{J}_m^+(x)$.

(GX3-3) $\tau_n^{-1}\sum_{i=1}^n|\gamma_{ni}\xi_{ni}| = O(1)$.

(GX4-1) F has a continuous and positive density f.

(GX4-2) f is uniformly continuous.

(GX4-3) The functions $J_m(x)$ and $J_m^+(x)$ are continuously differentiable with derivatives equal to $\dot{J}_m(x)$ and $\dot{J}_m^+(x)$.

(GX4-4) $|\dot{J}_m(x)| \vee \dot{J}_m^+(x) \to 0$ as $|x| \to |c| \vee |d|$.

(GX4-5) $\tau_n^{-1}\sum_{i=1}^n|\gamma_{ni}\xi_{ni}| = O(1)$.

4.2. Assumptions on $\boldsymbol{X}$.

(X.1) The first column of $\boldsymbol{X}$ is $\mathbf{1}$.

(X.2) $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ exists for all $n \geq p$.

(X.3) $n^{1/2}\max_i\|\boldsymbol{x}_{ni}'\boldsymbol{D}_{\boldsymbol{x}}^{-1}\| = O(1)$.

5. Formulae

6. Results

Lemma 6.1 (km1994)**.** *Under Assumption GX0, we have* $\forall x \in \psi$,

$$\sup_{x\in\bar{\psi}}\left|\tau_n^{-1}\sum_i \gamma_{ni}\{\mathbb{1}(\epsilon_i \leq x) - F(x)\} - \frac{J_m(x)}{m!}\tau_n^{-1}\sum_i \gamma_{ni}H_m(\eta_i)\right| = o_P(1).$$

Under Assumption GX1, we have $\forall x \in \psi$,

$$\Big|\tau_n^{-1}\sum_i \gamma_{ni}\{\mathbb{1}(\epsilon_i \le x+\xi_{ni})\} - F(x+\xi_{ni}) - \mathbb{1}(\epsilon_i \le x) + F(x)\}\Big| = o_P(1).$$

Under Assumptions GX1 and GX2, then

$$\Big|\tau_n^{-1}\sum_i \gamma_{ni}\{\mathbb{1}(\epsilon_i \le x+\xi_{ni})\} - \mathbb{1}(\epsilon_i \le x)\} - \tau_n^{-1}\sum_i \gamma_{ni}\xi_{ni} f(x)\Big| = o_P(1).$$

Under Assumptions GX1 and GX3, then for any $0<\kappa<\infty$ *with* $K=\{x\in\psi; |x|\le\kappa\}$,

$$\sup_{x\in K}\Big|\tau_n^{-1}\sum_i \gamma_{ni}\{\mathbb{1}(\epsilon_i \le x+\xi_{ni})\} - \mathbb{1}(\epsilon_i \le x)\} - \tau_n^{-1}\sum_i \gamma_{ni}\xi_{ni} f(x)\Big| = o_P(1).$$

Under Assumptions Assumptions GX1 and GX4, for $\bar{\psi}=[c,d]$,

$$\sup_{x\in\bar{\psi}}\Big|\tau_n^{-1}\sum_i \gamma_{ni}\{\mathbb{1}(\epsilon_i \le x+\xi_{ni})\} - \mathbb{1}(\epsilon_i \le x)\} - \tau_n^{-1}\sum_i \gamma_{ni}\xi_{ni} f(x)\Big| = o_P(1).$$

Lemma 6.2 (km1994). *Under Assumption X, we have*

$$\sup_{0\le\alpha\le 1}\Big\|B_x^{-1}T(\boldsymbol{\beta}(\alpha),\alpha) - S_x\frac{J_m(F^{-1}(\alpha))}{m!}\Big\| = o_P(1).$$

and

$$\|S_x\| = O_P(1).$$

Under Assumptions X and GX3-1, we have for every $0<K<\infty$ *and* $0<\alpha<1$,

$$\sup_{\|s\|\le K}\Big\|B_x^{-1}[\,T(\boldsymbol{\beta}(\alpha)+A_x^{-1}\boldsymbol{s},\alpha) - T(\boldsymbol{\beta}(\alpha),\alpha)\,] - sq(\alpha)\Big\| = o_P(1).$$

Also, for every $0<\alpha<1$,

$$\begin{aligned}A_x(\hat{\boldsymbol{\beta}}_{md}(\alpha) - \boldsymbol{\beta}(\alpha)) &= -\{q(\alpha)\}^{-1}B_x^{-1}T(\boldsymbol{\beta})(\alpha),\alpha) + o_P(1)\\ &= -\{q(\alpha)\}^{-1}S_x\frac{J_m(F^{-1}(\alpha))}{m!} + o_P(1),\end{aligned}$$

and

$$A_x\{\hat{\boldsymbol{\beta}}_n(\alpha) - \hat{\boldsymbol{\beta}}_{md}(\alpha)\} = o_P(1).$$

Under Assumptions X and GX4, we have for every $0<K<\infty$ *and* $0<\alpha<1$,

$$\sup_{0\le\alpha\le 1,\,\|s\|\le K}\Big\|B_x^{-1}[\,T(\boldsymbol{\beta}(\alpha)+A_x^{-1}\boldsymbol{s},\alpha) - T(\boldsymbol{\beta}(\alpha),\alpha)\,] - sq(\alpha)\Big\| = o_P(1).$$

Theorem 6.3. *Under Assumptions X and GX3-1, we have for any* $0<\alpha<1$,

$$\begin{aligned}A_x(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)) &= -\{q(\alpha)\}^{-1}B_x^{-1}T(\boldsymbol{\beta})(\alpha),\alpha) + o_P(1)\\ &= -\{q(\alpha)\}^{-1}S_x\frac{J_m(F^{-1}(\alpha))}{m!} + o_P(1).\end{aligned}$$

Consequently, for any $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < 1$,

$$\begin{aligned}&[A_x(\hat{\boldsymbol{\beta}}_n(\alpha_1) - \boldsymbol{\beta}(\alpha_1)), \ldots, A_x(\hat{\boldsymbol{\beta}}_n(\alpha_k) - \boldsymbol{\beta}(\alpha_k))] \\ &\qquad = -(m!)^{-1}[\{q(\alpha_1)\}^{-1} J_m(F^{-1}(\alpha_1)), \ldots, \{q(\alpha_k)\}^{-1} J_m(F^{-1}(\alpha_k))] \otimes S_x + o_P(1).\end{aligned}$$

Under Assumptions X and GX3-1 and GX3-2, then

$$A_x(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)) = O_P^*(1).$$