1. Reference

Fox and Taqqu (1986), AS.
Giraitis and Taqqu (1999), AS.

2. Notations

2.1. Notations.
1. $X_j$, $j \geq 1$ a stationary Gaussian sequence
2. $\mu$ mean
3. $\sigma^2 f(x, \theta)$ spectral density
4. $E \subset \mathbb{R}^p$ compact
5. $\bar{X}_N = (1/N) \sum_{j=1}^{N} X_j$
6. $Z = (X_1 - \bar{X}_N, \ldots, X_N - \bar{X}_N)'$
7. $A_N(\theta)$ $N \times N$ matrix with entries $[A_N(\theta)]_{jk} = a_{j-k}(\theta)$ below
8. $W(\theta)$ the $p \times p$ matrix with $j, k$th entry $w_{jk}(\theta)$
9. $\xi = (\xi_0, \ldots, \xi_r)$
10. $\phi = (\phi_0, \ldots, \phi_q)$
11. $\dot{G}$ the derivative of $G$

3. Fundamental Setting

(i) $\mu, \sigma^2 > 0$ and $\theta \in E$ are unknown parameters.
(ii) The periodogram $I_N(x)$ and $\tilde{I}_N(x)$

\[
I_N(x) = \frac{|\sum_{j=1}^{N} e^{ijx}(X_j - \bar{X}_N)|^2}{2\pi N}, \quad \tilde{I}_N(x) = \frac{|\sum_{j=1}^{N} e^{ijx}(X_j - \mu)|^2}{2\pi N}
\]

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(iii) Estimators $\tilde{\theta}_N$ and $\tilde{\sigma}^2_N$ are defined by

$$\left(\tilde{\theta}_N, \tilde{\sigma}^2_N\right) = \arg \max_{\theta, \sigma^2} \frac{1}{\sigma} \exp \left\{ -\frac{1}{2N\sigma^2} Z' A_N(\theta) Z \right\}.$$ 

Equivalently, $\tilde{\theta}_N$ minimizes

$$\sigma^2_N(\theta) = \frac{Z' A_N(\theta) Z}{N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f(x, \theta) \right]^{-1} I_N(x) dx.$$ 

(iv) $a_{j-k}(\theta)$ is defined by

$$a_j(\theta) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{ijx} f^{-1}(x, \theta) dx.$$ 

(v) $w_{jk}(\theta)$ is defined by

$$w_{jk}(\theta) = \int_{-\pi}^{\pi} f(x, \theta) \frac{\partial^2}{\partial \theta_j \partial \theta_k} f^{-1}(x, \theta) dx.$$ 

(vi) Fractional Gaussian noise

It is a stationary Gaussian sequence with mean 0 and covariance

$$r_k = EX_j X_{j+k} = C \left\{ |k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right\},$$

where $H$ is a parameter satisfying $\frac{1}{2} < H < 1$ and $C > 0$. This covariance satisfies

$$r_k \sim CH(2H - 1)k^{2H-2} \quad \text{as} \quad k \to \infty.$$ 

The spectral density $f(x, H)$ of fractional Gaussian noise is given by

$$f(x, H) = CF(H) f_0(x, H),$$

where

$$f_0(x, H) = (1 - \cos x) \sum_{k=-\infty}^{\infty} |x + 2k\pi|^{-1-2H}, \quad -\pi \leq x \leq \pi,$$

and

$$F(H) = \left\{ \int_{-\infty}^{\infty} (1 - \cos x)|x|^{-1-2H} dx \right\}^{-1}.$$ 

As $x \to 0$ we have

$$f(x, H) \sim \frac{CF(H)}{2} |x|^{-1+2H}.$$ 

(vii) Fractional ARMA
• \( g(x, \xi) = \sum_{j=0}^{p} \xi_j x^j \)
• \( h(x, \phi) = \sum_{j=0}^{q} \phi_j x^j \)
• \( g(x, \xi), h(x, \phi) \) have no zeros on the unit circle and no zeros in common.

A Gaussian sequence with mean 0 and spectral density \( f(x, d, \xi, \phi) \) is called a fractional ARMA process, where
\[
f(x, d, \xi, \phi) = C |e^{ix-1} - 2d| \frac{g(e^{ix}, \xi)}{h(e^{ix}, \phi)}|^2.
\]

Heuristically, it is the sequence which, when differenced \( d \) times, yields an ARMA process with spectral density
\[
C \left| \frac{g(e^{ix}, \xi)}{h(e^{ix}, \phi)} \right|^2.
\]

As \( x \to 0 \)
\[
f(x, d, \xi, \phi) \sim C \left| \frac{g(e^{ix}, \xi)}{h(e^{ix}, \phi)} \right|^2 |x|^{-2d}.
\]

(viii) If the mean is unknown
\[
I_N(x) = \frac{1}{2\pi} \left| \sum_{j=1}^{N} e^{ijx}(X_j - \bar{X}_N)^2 \right|.
\]

while if the mean is known
\[
\bar{I}_N(x) = \frac{1}{2\pi} \left| \sum_{j=1}^{N} e^{ijx}(X_j - \mu)^2 \right|.
\]

3.2. **Assumptions.** There exists \( 0 < \alpha(\theta) < 1 \) such that for each \( \delta > 0 \),
(A.1) \( g(\theta) = \int_{-\pi}^{\pi} \log f(x, \theta) dx \) can be differentiated twice under the integral sign.

(A.2) \( f(x, \theta) \) is continuous at all \( (x, \theta), x \neq 0 \), \( f^{-1}(x, \theta) \) is continuous at all \( (x, \theta) \), and
\[
f(x, \theta) = O(|x|^{-\alpha(\theta)-\delta}), \quad \text{as} \ x \to 0.
\]

(A.3) \( \partial / \partial \theta_j f^{-1}(x, \theta) \) and \( \partial^2 / \partial \theta_j \partial \theta_k f^{-1}(x, \theta) \) are continuous at all \( (x, \theta) \),
\[
\frac{\partial}{\partial \theta_j} f^{-1}(x, \theta) = O(|x|^{-\alpha(\theta)-\delta}) \quad \text{as} \ x \to 0, \quad 1 \leq j \leq p,
\]
and
\[
\frac{\partial^2}{\partial \theta_j \partial \theta_k} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-\delta}) \quad \text{as} \ x \to 0, \quad 1 \leq j, k \leq p.
\]
(A.4) \( \frac{\partial}{\partial x} f(x, \theta) \) is continuous at all \((x, \theta), x \neq 0, \) and 
\[
\frac{\partial}{\partial x} f(x, \theta) = O(|x|^{-\alpha(\theta)-1-\delta}), \quad \text{as } x \to 0.
\]

(A.5) \( \frac{\partial^2}{\partial x \partial \theta_j} f^{-1}(x, \theta) \) is continuous at all \((x, \theta), x \neq 0, \) and 
\[
\frac{\partial^2}{\partial x \partial \theta_j} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-1-\delta}) \quad \text{as } x \to 0, \quad 1 \leq j \leq p.
\]

(A.6) \( \frac{\partial^3}{\partial x^2 \partial \theta_j} f^{-1}(x, \theta) \) is continuous at all \((x, \theta), x \neq 0, \) and 
\[
\frac{\partial^3}{\partial x^2 \partial \theta_j} f^{-1}(x, \theta) = O(|x|^{\alpha(\theta)-2-\delta}) \quad \text{as } x \to 0, \quad 1 \leq j \leq p.
\]

4. Main Results

**Lemma 4.1.** If (A.1), (A.2) and (A.3) hold then 
\[
w_{jk}(\theta) = \int_{-\pi}^{\pi} \left( \frac{\partial}{\partial \theta_j} f^{-1}(x, \theta) \right) \left( \frac{\partial}{\partial \theta_k} f^{-1}(x, \theta) \right) f^2(x, \theta) dx.
\]

**Proof.** Note that 
\[
0 = \int_{-\pi}^{\pi} \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log f(x, \theta) dx \\
= - \int_{-\pi}^{\pi} \left( \frac{\partial}{\partial \theta_j} f^{-1}(x, \theta) \right) \left( \frac{\partial}{\partial \theta_k} f^{-1}(x, \theta) \right) f^2(x, \theta) dx + \int_{-\pi}^{\pi} f^{-1}(x, \theta) \frac{\partial^2}{\partial \theta_j \partial \theta_k} f(x, \theta) dx.
\]

(4.1) □

**Theorem 4.2** (ft (1986)). If \( f(x, \theta) \) satisfies conditions (A.2) and (A.4), then with probability 1, 
\[
\lim_{N \to \infty} \tilde{\theta}_N = \theta_0.
\]
\[
\lim_{N \to \infty} \tilde{\sigma}_N^2 = \sigma_0^2.
\]

(\text{thm1:ft1986})

**Theorem 4.3** (ft (1986)). If conditions (A.1)-(A.6) are satisfied then the random vector \( \sqrt{N}(\tilde{\theta}_N - \theta_0) \) tends in distribution to a normal random vector with mean 0 and covariance matrix \( 4\pi W^{-1}(\theta_0) \).

**Proof.** The proof is given by the following 3 steps.
Step 1. Set \( m_{N,j} = E \frac{\partial}{\partial \theta_j} \sigma_N^2(\theta_0) \) with
\[
Y_N = \sum_{j=1}^{p} c_j \left[ \frac{\partial}{\partial \theta_j} \sigma_N^2(\theta_0) - m_{N,j} \right].
\]
As a result,
\[
\sqrt{N} Y_N \xrightarrow{d} \mathcal{N}(0, s^2).
\]

Step 2. From Lemma 8.1 of Fox and Taqqu (1983),
\[
\lim_{N \to \infty} \sqrt{N} (m_{N,l} - \mu_{N,l}) = 0.
\]

Step 3. Set
\[
\mu_{N,l} = E \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta_l} f^{-1}(x, \theta_0) \hat{I}_N(x) dx \right\}.
\]
As a result,
\[
\lim_{N \to \infty} \sqrt{N} \mu_{N,l} = 0, \quad l = 1, \ldots, p.
\]

\((\text{thm3:ft1986})\)

Theorem 4.4. The conclusions of Theorems 4.2 and 4.3 hold if \( X_j - \mu \) is fractional Gaussian noise with \( \frac{1}{2} < H < 1 \) or a fractional ARMA process with \( 0 < d < \frac{1}{2} \).

Corollary 4.5. When \( \mu = EX_j \) is known, Theorems 4.2, 4.3 and 4.4 hold if \( I_N(x) \) is replaced by \( \hat{I}_N(x) \).

5. Further Reading

- See Sinai (1976, TPA) for the derivation of the spectral density of long-range dependent process.
- See Granger and Joyeux (1980) and Hosking (1981) for the modeling of strongly dependent phenomena.
- Fox and Taqqu (1983), technical report 590, Cornell Univ.
- Fox and Taqqu (1985), AP.
6. NOTATIONS

1. $X_t$, $t \in \mathbb{Z}$ a strongly dependent time series
2. $f(x)$ the spectral density of the time series
3. $L$ a slowly varying function at infinity
4. $\alpha$ the exponent
5. $G$ a polynomial
6. $Y_t = G(X_t)$
7. $s_\theta(x) = \sigma^2 g_\theta(x)$ the spectral density of the time series $Y_t$
8. $L_{G,\theta}$ a slowly varying function
9. $(\theta, \sigma^2)$ the parameters
10. $A_{N,\theta} = \{a_\theta(t - s)\}_{t, s = 1, \ldots, N}$
11. $\rho_1 = 2 \sum_{t \in \mathbb{Z}} [E\hat{G}(X_t)G(X_0)] \nabla a_{\theta_0}(t)$

7. FUNDAMENTAL SETTING


(i) Estimators

$$\hat{\theta}_N = \arg \min_{\theta} N^{-1} Y' A_{N,\theta} Y,$$

where $Y = (Y_1, \ldots, Y_N)$.

(ii) $a_\theta(t)$ is defined by

$$a_\theta(t) = \int_{-\pi}^{\pi} e^{its} g_\theta^{-1}(x) dx.$$

(iii) $v_{m,n}(t - s)$

$$v_{m,n}(t - s) = \frac{1}{m!n!} [E G_m^m(X_t) G_n^n(X_s)] \nabla a_{\theta_0}(t - s).$$

(iv) $\rho_k$

$$\rho_k = \sum_{m,n \geq 0; m+n=k} \sum_{t \in \mathbb{Z}} v_{m,n}(t)$$

7.2. Assumptions.

(i) The spectral density $f(x)$ satisfies

$$f(x) = |x|^{-\alpha} L(1/|x|), \quad x \in [-\pi, \pi], \quad (0 < \alpha < 1).$$

Remark 7.1. Note that $\alpha = 1 - 2H$ ($1/2 < H < 1$).

(ii) $g_\theta$ satisfies

$$g_\theta(x) = |x|^{-\alpha_G(\theta)} L_{G,\theta}(1/|x|), \quad |x| \leq \pi,$$

where $0 \leq \alpha_G(\theta) < 1$. 

(iii) Suppose
\[ \int_{-\pi}^{\pi} \log g_{\theta}(x) dx = 0, \quad \theta \in \Theta. \] (7.1)

(iv) \((\partial^2/\partial \theta_i \partial \theta_j)g_{\theta}^{-1}(x)\) is a continuous function in \((x, \theta)\).

(v) For any small fixed number \(\epsilon > 0\),
\[
\begin{align*}
\left| \frac{\partial}{\partial \theta_j} g_{\theta}^{-1}(x) \right| &\leq C|x|^\alpha_G - \epsilon, \quad |x| \leq \pi \quad \text{for} \quad \theta = \theta_0, \\
\left| \frac{\partial^2}{\partial x \partial \theta_j} g_{\theta}^{-1}(x) \right| &\leq C|x|^\alpha_G - 1 - \epsilon, \quad |x| \leq \pi \quad \text{for} \quad \theta = \theta_0.
\end{align*}
\]

(vi) the spectral density \(f\) of the Gaussian sequence \((X_t)\) satisfies
\[
\left| \frac{d}{dx} f(x) \right| \leq C|x|^{-\alpha_{\epsilon}}, \quad |x| \leq \pi,
\]
where \(\epsilon = \epsilon(\theta) > 0\) is any fixed number.

8. Main Results

**Theorem 8.1.** Assume that (7.1) holds and that \(g_{\theta}^{-1}(x)\) is a continuous function. Then almost surely,
\[
\lim_{N \to \infty} \hat{\theta}_N = \theta_0.
\]
\[
\lim_{N \to \infty} \hat{\sigma}_N^2 = \sigma_0^2.
\]

**Theorem 8.2.** Suppose that Assumptions 7.2 hold, that \(W_{\theta_0}^{-1}\) exists and \(\rho_1 \neq 0\). Then
\[
\hat{\theta}_N - \theta_0 = -(2\pi \sigma_0^2)^{-1}W_{\theta_0}^{-1}\rho_1 \left( (N^{-1} \sum_{j=1}^{N} X_j) (1 + o_P(1)) \right).
\]

**Corollary 8.3.** Theorem 8.2 implies that
\[
[N^{1-\alpha} \Lambda^{-1}(N)]^{1/2} (\hat{\theta}_N - \theta_0) \xrightarrow{d} (2\pi \sigma_0^2)^{-1}W_{\theta_0}^{-1}\rho_1 \xi,
\]
where \(\xi\) is a Gaussian random variable with zero mean and variance \(E\xi^2 = 2/(\alpha(\alpha + 1))\).

**Example 1.** In the case of \(G(X_t) = X_t, \hat{G}(X_t) = 1\) and \(E\hat{G}(X_t)G(X_t) = EX_t = 0\) and therefore \(\rho_1 = 0\).

**Theorem 8.4.** Let \(\rho_1 = 0, \rho_2 \neq 0\).
(i) If $1/2 < \alpha < 1$, then
\[ N^{(1-\alpha)}L^{-1}(N)(\hat{\theta}_N - \theta_0) \xrightarrow{L} (2\pi\sigma_0^2)^{-1}W_{\theta_0}^{-1}\rho_2 I_2, \]
where $I_2$ has the Rosenblatt distribution, i.e.,
\[ I_2 = \int_{\mathbb{R}^2} \frac{\exp(it(x_1 + x_2)) - 1}{i(x_1 + x_2)}|x_1|^{-\alpha}|x_2|^{-\alpha}Z(dx_1)Z(dx_2), \quad \alpha > 1/2. \]

(ii) If $0 < \alpha < 1/2$, then
\[ \sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{L} N(0, (2\pi\sigma_0^2)^{-2}W_{\theta_0}^{-1}DW_{\theta_0}^{-1}), \]
where $D$ is a $p \times p$ matrix with entries
\[ d(i, j) = \sum_{t \in \mathbb{Z}} \left[ \sum_{s_1, s_2 \in \mathbb{Z}} \hat{\theta}_0^{(i)}(s_1)\hat{\theta}_0^{(j)}(s_2)\Cov(G(X_t)G(X_{t+s_1}), G(X_0)G(X_{s_2})) \right]. \]

9. WORDS

1. rather the exception than the rule どちらかといえば例外的で

10. NEW KNOWLEDGE

- The compensation effect in the Whittle estimator appears when the observations $X_t$ are pure Gaussian or linear is rather the exception than the rule!!