LONG RANGE DEPENDENCE

YAN LIU

1. Reference

Fox and Taqqu (1986), AS.
Giraitis and Taqqu (1999), AS.

2. notations

2.1. Notations.

1. X_j, $j \geq 1$	a stationary Gaussian sequence
2. μ	mean
3. $\sigma^2 f(x, \theta)$	spectral density
4. $E \subset \mathbb{R}^p$	compact
5. $\bar{X}_N$	$= (1/N) \sum_{j=1}^{n} X_j$
6. $\boldsymbol{Z}$	$= (X_1 - \bar{X}_N, \ldots, X_N - \bar{X}_N)'$
7. $A_N(\theta)$	$N \times N$ matrix with entries $[A_N(\theta)]_{jk} = a_{j-k}(\theta)$ below
8. $W(\theta)$	the $p \times p$ matrix with j, kth entry $w_{jk}(\theta)$
9. ξ	$= (\xi_0 \ldots, \xi_r)$
10. ϕ	$= (\phi_0, \ldots, \phi_q)$
11. $\dot{G}$	the derivative of G

3. Fundamental Setting

3.1. Basics.

(i) μ, $\sigma^2 > 0$ and $\theta \in E$ are unknown parameters.

(ii) The periodogram $I_N(x)$ and $\tilde{I}_N(x)$

$$I_N(x) = \frac{|\sum_{j=1}^{N} e^{ijx}(X_j - \bar{X}_N)|^2}{2\pi N}, \qquad \tilde{I}_N(x) = \frac{|\sum_{j=1}^{N} e^{ijx}(X_j - \mu)|^2}{2\pi N}$$

Date: November 29, 2014.

(iii) Estimators $\bar{\theta}_N$ and $\bar{\sigma^2}_N$ are defined by

$$(\bar{\theta}_N, \bar{\sigma}_N^2) = \arg\max_{\theta,\sigma^2} \frac{1}{\sigma} \exp\left\{ -\frac{1}{2N\sigma^2} Z' A_N(\theta) Z \right\}.$$

Equivalently, $\bar{\theta}_N$ minimizes

$$\sigma_N^2(\theta) = \frac{Z' A_N(\theta) Z}{N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x,\theta)]^{-1} I_N(x) dx.$$

(iv) $a_{j-k}(\theta)$ is defined by

$$a_j(\theta) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{ijx} f^{-1}(x,\theta) dx.$$

(v) $w_{jk}(\theta)$ is defined by

$$w_{jk}(\theta) = \int_{-\pi}^{\pi} f(x,\theta) \frac{\partial^2}{\partial\theta_j \partial\theta_k} f^{-1}(x,\theta) dx.$$

(vi) Fractional Gaussian noise

It is a stationary Gaussian sequence with mean 0 and covariance

$$r_k = EX_j X_{j+k} = \frac{C}{2}\{|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\},$$

where H is a parameter satisfying $\frac{1}{2} < H < 1$ and $C > 0$. This covariance satisfies

$$r_k \sim CH(2H-1)k^{2H-2} \quad \text{as } k \to \infty.$$

The spectral density $f(x,H)$ of fractional Gaussian noise is given by

$$f(x,H) = CF(H) f_0(x,H),$$

where

$$f_0(x,H) = (1-\cos x) \sum_{k=-\infty}^{\infty} |x + 2k\pi|^{-1-2H}, \quad -\pi \le x \le \pi,$$

and

$$F(H) = \left\{ \int_{-\infty}^{\infty} (1-\cos x)|x|^{-1-2H} dx \right\}^{-1}$$

As $x \to 0$ we have

$$f(x,H) \sim \frac{CF(H)}{2} |x|^{1-2H}.$$

(vii) Fractional ARMA

- $g(x,\xi) = \sum_{j=0}^{p} \xi_j x^j$
- $h(x,\phi) = \sum_{j=0}^{q} \phi_j x^j$
- $g(x,\xi), h(x,\phi)$ have no zeros on the unit circle and no zeros in common.

A Gaussian sequence with mean 0 and spectral density $f(x,d,\xi,\phi)$ is called a fractional ARMA process, where

$$f(x,d,\xi,\phi) = C|e^{ix-1}|^{-2d}\Big|\frac{g(e^{ix},\xi)}{h(e^{ix},\phi)}\Big|^2,$$

Heuristically, it is the sequence which, when differenced d times, yields an ARMA process with spectral density

$$C\Big|\frac{g(e^{ix},\xi)}{h(e^{ix},\phi)}\Big|^2.$$

As $x \to 0$

$$f(x,d,\xi,\phi) \sim C\Big|\frac{g(e^{ix},\xi)}{h(e^{ix},\phi)}\Big|^2 |x|^{-2d}.$$

(viii) If the mean is unknown

$$I_N(x) = \frac{1}{2\pi N}\Big|\sum_{j=1}^{N} e^{ijx}(X_j - \bar{X}_N)^2\Big|,$$

while if the mean is known

$$\tilde{I}_N(x) = \frac{1}{2\pi N}\Big|\sum_{j=1}^{N} e^{ijx}(X_j - \mu)^2\Big|.$$

3.2. **Assumptions.** There exists $0 < \alpha(\theta) < 1$ such that for each $\delta > 0$,
(A.1) $g(\theta) = \int_{-\pi}^{\pi} \log f(x,\theta)dx$ can be differentiated twice under the integral sign.

(A.2) $f(x,\theta)$ is continuous at all (x,θ), $x \neq 0$, $f^{-1}(x,\theta)$ is continuous at all (x,θ), and

$$f(x,\theta) = O(|x|^{-\alpha(\theta)-\delta}), \quad \text{as } x \to 0.$$

(A.3) $\partial/\partial\theta_j f^{-1}(x,\theta)$ and $\partial^2/\partial\theta_j\partial\theta_k f^{-1}(x,\theta)$ are continuous at all (x,θ),

$$\frac{\theta}{\partial\theta_j} f^{-1}(x,\theta) = O(|x|^{-\alpha(\theta)-\delta}) \quad \text{as } x \to 0, \quad 1 \leq j \leq p,$$

and

$$\frac{\partial^2}{\partial\theta_j\partial\theta_k} f^{-1}(x,\theta) = O(|x|^{\alpha(\theta)-\delta}) \quad \text{as } x \to 0, \quad 1 \leq j,k \leq p.$$

(A.4) $\partial/\partial x f(x,\theta)$ is continuous at all (x,θ), $x \neq 0$, and

$$\frac{\partial}{\partial x} f(x,\theta) = O(|x|^{-\alpha(\theta)-1-\delta}), \quad \text{as } x \to 0.$$

(A.5) $\partial^2/\partial x \partial \theta_j f^{-1}(x,\theta)$ is continuous at all (x,θ), $x \neq 0$, and

$$\frac{\partial^2}{\partial x \partial \theta_j} f^{-1}(x,\theta) = O(|x|^{\alpha(\theta)-1-\delta}) \quad \text{as } x \to 0, \quad 1 \leq j \leq p.$$

(A.6) $\partial^3/\partial x^2 \partial \theta_j f^{-1}(x,\theta)$ is continuous at all (x,θ), $x \neq 0$, and

$$\frac{\partial^3}{\partial^2 x \partial \theta_j} f^{-1}(x,\theta) = O(|x|^{\alpha(\theta)-2-\delta}) \quad \text{as } x \to 0, \quad 1 \leq j \leq p.$$

4. Main Results

Lemma 4.1. *If (A.1), (A.2) and (A.3) hold then*

$$w_{jk}(\theta) = \int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \theta_j} f^{-1}(x,\theta) \right) \left(\frac{\partial}{\partial \theta_k} f^{-1}(x,\theta) \right) f^2(x,\theta) dx.$$

Proof. Note that

$$\begin{aligned} 0 &= \int_{-\pi}^{\pi} \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log f(x,\theta) dx \\ &= -\int_{-\pi}^{\pi} \left(\frac{\partial}{\partial \theta_j} f^{-1}(x,\theta) \right) \left(\frac{\partial}{\partial \theta_k} f^{-1}(x,\theta) \right) f^2(x,\theta) dx + \int_{-\pi}^{\pi} f^{-1}(x,\theta) \frac{\partial^2}{\partial \theta_j \partial \theta_k} f(x,\theta) dx. \end{aligned} \tag{4.1}$$

{?}

□

⟨thm1:ft1986⟩ **Theorem 4.2** (ft (1986))**.** *If $f(x,\theta)$ satisfies conditions (A.2) and (A.4), then with probability 1,*

$$\begin{aligned} \lim_{N\to\infty} \bar{\theta}_N &= \theta_0. \\ \lim_{N\to\infty} \bar{\sigma}^2_N &= \sigma^2_0. \end{aligned}$$

⟨thm2:ft1986⟩ **Theorem 4.3** (ft (1986))**.** *If conditions (A.1)-(A.6) are satisfied then the random vector $\sqrt{N}(\bar{\theta}_N - \theta_0)$ tends in distribution to a normal random vector with mean 0 and covariance matrix $4\pi W^{-1}(\theta_0)$.*

Proof. The proof is given by the following 3 steps.

Step 1. Set $m_{N,j} = E\,\partial/\partial\theta_j \sigma_N^2(\theta_0)$ with

$$Y_N = \sum_{j=1}^{p} c_j\Big[\frac{\partial}{\partial\theta_j}\sigma_N^2(\theta_0) - m_{N,j}\Big].$$

As a result,

$$\sqrt{N}Y_N \xrightarrow{\mathcal{L}} \mathcal{N}(0,s^2).$$

Step 2. From Lemma 8.1 of Fox and Taqqu (1983),

$$\lim_{N\to\infty}\sqrt{N}(m_{N,l}-\mu_{N,l}) = 0.$$

Step 3. Set

$$\mu_{N,l} = E\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta_l}f^{-1}(x,\theta_0)\tilde{I}_N(x)dx\Big\}.$$

As a result,

$$\lim_{N\to\infty}\sqrt{N}\mu_{N,l} = 0, \quad l=1,\dots,p.$$

□

⟨thm3:ft1986⟩ **Theorem 4.4.** *The conclusions of Theorems 4.2 and 4.3 hold if $X_j - \mu$ is fractional Gaussian noise with $\frac{1}{2} < H < 1$ or a fractional ARMA process with $0 < d < \frac{1}{2}$.*

Corollary 4.5. When $\mu = EX_j$ is known, Theorems 4.2, 4.3 and 4.4 hold if $I_N(x)$ is replaced by $\tilde{I}_N(x)$.

5. Further Reading

- See Sinai (1976, TPA) for the derivation of the spectral density of long-range dependent process
- See Granger and Joyeux (1980) and Hosking (1981) for the modeling of strongly dependent phenomena.
- Fox and Taqqu (1983), technical report 590, Cornell Univ.
- Fox and Taqqu (1985), AP

6. NOTATIONS

1. X_t, $t \in \mathbb{Z}$	a strongly dependent time series
2. $f(x)$	the spectral density of the time series
3. L	a slowly varying function at infinity
4. α	the exponent
5. G	a polynomial
6. Y_t	$= G(X_t)$
7. $s_\theta(x) = \sigma^2 g_\theta(x)$	the spectral density of the process Y_t
8. $L_{G,\theta}$	a slowly varying function
9. (θ, σ^2)	the parameters
10. $A_{N,\theta}$	$= \{a_\theta(t-s)\}_{t,s=1,\ldots,N}$
11. ρ_1	$= 2\sum_{t\in\mathbb{Z}}[E\dot{G}(X_t)G(X_0)]\nabla a_{\theta_0}(t)$

7. FUNDAMENTAL SETTING

7.1. Basics.

(i) Estimators

$$\hat{\theta}_N = \arg\min_\theta N^{-1} Y' A_{N,\theta} Y,$$

where $Y = (Y_1, \ldots, Y_N)$.

(ii) $a_\theta(t)$ is defined by

$$a_\theta(t) = \int_{-\pi}^{\pi} e^{its} g_\theta^{-1}(x)dx.$$

(iii) $v_{m,n}(t-s)$

$$v_{m,n}(t-s) = \frac{1}{m!n!}[EG^m(X_t)G^{(n)}(X_s)]\nabla a_{\theta_0}(t-s).$$

(iv) ρ_k

$$\rho_k = \sum_{m,n\geq 0;\, m+n=k} \sum_{t\in\mathbb{Z}} v_{m,n}(t)$$

⟨ss:7.2⟩ 7.2. **Assumptions.**

(i) The spectral density $f(x)$ satisfies

$$f(x) = |x|^{-\alpha} L(1/|x|), \quad x \in [-\pi, \pi], \quad (0 < \alpha < 1).$$

Remark 7.1. Note that $\alpha = 1 - 2H$ $(1/2 < H < 1)$.

(ii) g_θ satisfies

$$g_\theta(x) = |x|^{-\alpha_G(\theta)} L_{G,\theta}(1/|x|), \quad |x| \leq \pi,$$

where $0 \leq \alpha_G(\theta) < 1$.

(iii) Suppose

$$\int_{-\pi}^{\pi} \log g_\theta(x)dx = 0, \quad \theta \in \Theta. \tag{7.1}$$ eq2.3:gt1999

(iv) $(\partial^2/\partial\theta_i\partial\theta_j)g_\theta^{-1}(x)$ is a continuous function in (x,θ).

(v) For any small fixed number $\epsilon > 0$,

$$\begin{aligned}\left|\frac{\partial}{\partial\theta_j}g_\theta^{-1}(x)\right| &\le C|x|^{\alpha_G(\theta)-\epsilon}, \quad |x| \le \pi \quad \text{for } \theta = \theta_0,\\ \left|\frac{\partial^2}{\partial x\partial\theta_j}g_\theta^{-1}(x)\right| &\le C|x|^{\alpha_G(\theta)-1-\epsilon}, \quad |x| \le \pi \quad \text{for } \theta = \theta_0.\end{aligned}$$

(vi) the spectral density f of the Gaussian sequence (X_t) satisfies

$$\left|\frac{d}{dx}f(x)\right| \le C|x|^{-\alpha-1\epsilon}, \quad |x| \le \pi,$$

where $\epsilon = \epsilon(\theta) > 0$ is any fixed number.

8. Main Results

Theorem 8.1. *Assume that* (7.1) *holds and that $g_\theta^{-1}(x)$ is a continuous function. Then almost surely,*

$$\begin{aligned}\lim_{N\to\infty} \hat\theta_N &= \theta_0.\\ \lim_{N\to\infty} \hat\sigma_N^2 &= \sigma_0^2.\end{aligned}$$

⟨thm2.2:gt1999⟩ **Theorem 8.2.** *Suppose that Assumptions 7.2 hold, that $W_{\theta_0}^{-1}$ exists and $\rho_1 \ne 0$. Then*

$$\hat\theta_N - \theta_0 = -(2\pi\sigma_0^2)^{-1}W_{\theta_0}^{-1}\rho_1\Big(N^{-1}\sum_{j=1}^{N}X_j\Big)(1+o_P(1)).$$

Corollary 8.3. Theorem 8.2 implies that

$$[N^{1-\alpha}L^{-1}(N)]^{1/2}(\hat\theta_N - \theta_0) \xrightarrow{\mathcal{L}} (2\pi\sigma_0^2)^{-1}W_{\theta_0}^{-1}\rho_1\xi,$$

where ξ is a Gaussian random variable with zero mean and variance $E\xi^2 = 2/(\alpha(\alpha+1))$.

Example 1. In the case of $G(X_t) = X_t$, $\dot G(X_t) = 1$ and $E\dot G(X_t)G(X_t) = EX_t = 0$ and therefore $\rho_1 = 0$.

Theorem 8.4. *Let $\rho_1 = 0$, $\rho_2 \ne 0$.*

(i) If $1/2 < \alpha < 1$, *then*

$$N^{(1-\alpha)}L^{-1}(N)(\hat{\theta}_N - \theta_0) \xrightarrow{\mathcal{L}} (2\pi\sigma_0^2)^{-1}W_{\theta_0}^{-1}\rho_2 I_2,$$

where I_2 *has the Rosenblatt distribution, i.e.,*

$$I_2 = \int_{\mathbb{R}^2} \frac{\exp(it(x_1+x_2))-1}{i(x_1+x_2)}|x_1|^{-\alpha}|x_2|^{-\alpha}Z(dx_1)Z(dx_2), \quad \alpha > 1/2.$$

(ii) If $0 < \alpha < 1/2$, *then*

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (2\pi\sigma_0^2)^{-2}W_{\theta_0}^{-1}DW_{\theta_0}^{-1}),$$

where D *is a* $p \times p$ *matrix with entries*

$$d(i,j) = \sum_{t\in\mathbb{Z}}\left[\sum_{s_1,s_2\in\mathbb{Z}} \dot{a}_{\theta_0}^{(i)}(s_1)\dot{a}_{\theta_0}^{(j)}(s_2)\mathrm{Cov}(G(X_t)G(X_{t+s_1}), G(X_0)G(X_{s_2}))\right].$$

9. Words

1. rather the exception than the rule　どちらかといえば例外的で

10. New knowledge

- The compensation effect in the Whittle estimator appears when the observations X_t are pure Gaussian or linear is rather the exception than the rule!!