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Asymptotic Theory in Statistics

Chapter 1

Stable Random Variables

Before introducing the stable random variables, we have to understand the scope where the stable random variables dominate. The crucial concept here is "in the domain of attraction".

1.1 Domain of Attraction

ϵ is in the domain of attraction of a stable law with a parameter α and write $\epsilon \in \mathcal{D}(\alpha)$ if

$$P(\epsilon > x) = c_1 x^{-\alpha} L(x)(1 + \alpha_1(x)), \quad x > 0, c_1 \geq 0, \tag{1.1.1}$$

and

$$P(\epsilon < -x) = c_2 x^{-\alpha} L(x)(1 + \alpha_2(x)), \quad x > 0, c_2 \geq 0, \tag{1.1.2}$$

with $0 < \alpha < 2$, $L(x)$ a slowly varying function at ∞ and $\alpha_i(x) \to 0$ as $|x| \to \infty$. If $L(x) = 1$, then ϵ is in the normal domain of attraction of a stable law with parameter α.

Another form for the random variables in the domain of attraction with distribution function F satisfies

$$\begin{cases} x^\alpha P(\epsilon > x) = x^\alpha (1 - F(x)) \to pC, & x > 0, \\ x^\alpha P(\epsilon < -x) = x^\alpha F(-x) \to qC, & x > 0, \end{cases} \tag{1.1.3}$$

which means

$$\begin{cases} c_1 = pC, \\ c_2 = qC. \end{cases} \tag{1.1.4}$$

1.2 Parametrization of stable distributions

Let Y be distributed as stable distribution $S_\alpha(\sigma, \beta, \mu)$, then its characteristic function is

$$E(e^{itY}) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta (\text{sign} t) \tan \frac{\pi\alpha}{2}) + i\mu t\} & \alpha \neq 1, \\ \exp\{-\sigma |t| (1 + \frac{2i\beta}{\pi} (\text{sign} t) \log|t|) + i\mu t\}, \end{cases} \tag{1.2.1}$$

where σ is the scale parameter, β is the skewness parameter and μ is the location parameter.

Stable random variables has an exact form of their tails, that is,

$$\begin{cases} x^\alpha P(\epsilon > x) = \frac{1+\beta}{2}\sigma^\alpha C_\alpha, & x > 0, \\ x^\alpha P(\epsilon < -x) = \frac{1-\beta}{2}\sigma^\alpha C_\alpha, & x > 0. \end{cases} \tag{1.2.2}$$

Here C_α is a constant depending on α, and

$$C_\alpha = \begin{cases} \frac{1}{\Gamma(1-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} \quad \text{if } \alpha = 1. \end{cases} \tag{1.2.3}$$

Since $0 < \alpha < 2$ and $\Gamma(2-\alpha) = (1-\alpha)\Gamma(1-\alpha)$,

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} \quad \text{if } \alpha = 1. \end{cases} \tag{1.2.4}$$

is used in some books.

1.3 Klüppelberg and Mikosch (1995)

$$C_\alpha = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \quad \text{if } \alpha = 1 \end{cases} \tag{1.3.1}$$

を α に対する定数とする。Z_t の極限分布をこの Klüppelberg and Mikosch (1995) の極限分布の結果を参考にすれば、我々の結果である極限分布 $(Z_1/Y_0)^2$ の Z_1 と Y_0 はそれぞれ、

$$Y_0 \quad =_d \quad S_{\alpha/2}(C_{\alpha/2}^{-2/\alpha}, 1, 0); \tag{1.3.2}$$

$$Z_1 \quad =_d \quad S_\alpha(C_\alpha^{1/\alpha}, 0, 0). \tag{1.3.3}$$

この結果はかなり引用されていますが、果たして正しいのでしょうか。

1.4 Davis and Brockwell (1991)

少し違う形で Davis and Brockwell も結果を与えてくれている (Sec 13.3)。ただし、

$$C_\alpha = \begin{cases} \frac{\sigma}{\Gamma(1-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1 \\ \frac{2\sigma}{\pi} \quad \text{if } \alpha = 1 \end{cases} \tag{1.4.1}$$

Davis and Brockwell の結果は以下である:

$$Y_0 \quad =_d \quad S_{\alpha/2}((C_\alpha\Gamma(1-\alpha)\cos(\pi\alpha/2))^{-2/\alpha}, 1, 0); \tag{1.4.2}$$

$$Z_1 \quad =_d \quad S_\alpha((C_\alpha^2\Gamma(1-\alpha)\cos(\pi\alpha/2))^{1/\alpha}, 0, 0). \tag{1.4.3}$$

Klüppelberg and Mikosch (1995) の結果と異なるように見えるが、半分正しい。C_α の定義によって形がわからなくなっているからだ。ちょっと確認してみる。Z_1 について、

$$C_\alpha^2\Gamma(1-\alpha) = C_\alpha * d. \tag{1.4.4}$$

1.5 Point Process

1.5.1 Notations 1

Let $\mathcal{F}_s$ be the collection of step functions on $\mathbb{R} - \{0\}$ with bounded support.

1.5.2 Conditions

On sequence

(a1) $\{X_k\}$ is an i.i.d sequence of random variables.

(a2) $\{X_k\}$ is a strictly stationary sequence of random variables.

On random variables

The conditions (b1) on random variables are:

$$P(|X_k| > x) = x^{-\alpha} L(x) \tag{1.5.1}$$

with $\alpha \in (0, 2)$ and $L(x)$ a slowly varying function at ∞;

$$\frac{P(X_k > x)}{P(|X_k| > x)} \to p, \quad \frac{P(X_k < -x)}{P(|X_k| > x)} \to q \tag{1.5.2}$$

as $x \to \infty$, $0 \le p \le 1$ and $q = 1 - p$.

On technical conditions

(c1) Let a_n be defined as

$$a_n = \inf\{\, x : P(|X_1| > x) \le n^{-1} \,\}. \tag{1.5.3}$$

(d1) The mixing condition is defined as

$$E \exp\left(-\sum_{j=1}^{n} f(X_j/a_n)\right) - \left(E \exp\left(-\sum_{j=1}^{r_n} f(X_j/a_n)\right)\right)^{[n/r_n]} \to 0 \tag{1.5.4}$$

as $n \to \infty$ for all $f \in \mathcal{F}_s$.

1.5.3 Explanation

- (a1) and (b1) are necessary and sufficient for the existence of normalizing constants a_n, b_n for which $(S_n - b_n)/a_n$ converges weakly to some stable law with index α (cf. Feller (1971)). They also imply that

  $$\lim_{n\to\infty} \frac{P(S_n > t_n)}{nP(X_1 > t_n)} = 1 \tag{1.5.5}$$

  for any constants t_n satisfying $nP(X_1 > t_n) \to 0$.

- (a1), (b1) and (c1) imply that

$$nP(|X_1| > a_n x) \to x^{-\alpha} \quad \text{for all } x > 0. \tag{1.5.6}$$

Or we can write it more implicitly,

$$nP(X_1/a_n \in \cdot) \to_v \mu(\cdot), \tag{1.5.7}$$

where μ is the measure

$$\mu(dx) = \lambda(dx) = \alpha p x^{-\alpha-1} 1_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)dx, \tag{1.5.8}$$

and $\to_v$ denotes vague convergence on $\mathbb{R} - \{0\}$.

1.5.4 Notations 2

Define the point process

$$N_n = \sum_{j=1}^{n} \delta_{X_j/a_n}, \tag{1.5.9}$$

where δ_x represents unit point measure at the point x.

For any $y \geq 0$, define

$$M_y = \{\mu \in M : \mu([-y,y]^c) > 0 \text{ and } \mu([-x,x]^c) = 0 \text{ for some } 0 < x(= x_\mu) < \infty\}. \tag{1.5.10}$$

For $\mu \in M_0$, let $\mu_+ = \max(0, \text{largest point of } \mu)$, $\mu_- = \min(0, \text{smallest point of } \mu)$ and $x_\mu = \max(\mu_+, \mu_-)$. Define a mapping on M_0 by

$$\Omega : \mu \to (x_\mu, \mu(x_\mu \cdot)). \tag{1.5.11}$$

The mapping Ω is continuous with range $(0,\infty) \times \tilde{M}$, where $\tilde{M} = \{\mu \in M : \mu([-1,1]^c) = 0,\ \mu(\{-1\} \cup \{1\}) > 0\}$. Denote by $\mathcal{B}(\tilde{M})$ the Borel σ-field of $\tilde{M}$.

$\gamma := \lambda\{\mu : \mu([-1,1]^c) > 0\} \in (0,1]$

$$X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}. \tag{1.5.12}$$

$$\tilde{a}_n = \inf\{\, x : P(|Z_0 Z_1| > x) \leq n^{-1} \,\}. \tag{1.5.13}$$

$$\sum_{j=0}^{\infty} |c_j|^\delta < \infty \quad \text{for some} \quad \delta < \alpha,\ \delta \leq 1. \tag{1.5.14}$$

$$\sum_{j=-\infty}^{\infty} |c_j|^\delta |j| < \infty \quad \text{with} \quad \begin{cases} \delta = 1, & \text{if } \alpha > 1 \\ 0 < \delta < \alpha & \text{if } \alpha \leq 1. \end{cases} \tag{1.5.15}$$

$$\hat{\rho}(h) = \frac{C(h)}{C(0)}, \quad h \geq 0, \tag{1.5.16}$$

where

$$C(h) = \sum_{t=1}^{n} X_t X_{t+h}. \tag{1.5.17}$$

Further,

$$\rho(h) = \frac{\sum_j c_j c_{j+h}}{\sum_j c_j^2}.$$

1.5.5 Theorems

Theorem 1.5.1 (DH1995:2.3 [?]). *Assume that the condition (d1) holds for $\{X_j\}$, and $N_n \to_d$ some $N \neq o$. Then N is infinitely divisible with canonical measure λ satisfying $\lambda(M_0^c) = 0$ and $\lambda \circ \Omega^{-1} = \nu \times \mathcal{O}$, where $\mathcal{O}$ is a probability measure on $(M, \mathcal{B}(\tilde{M}))$, and*

$$\nu(dy) = \gamma\alpha y^{-\alpha-1} I_{(0,\infty)}(y)\, dy. \tag{1.5.18}$$

In this case the Laplace transform of N is

$$\exp\left\{-\int_0^\infty \int_{\tilde{M}} (1 - \exp(-\mu f(y\cdot)))\mathcal{O}(d\mu)\nu(dy)\right\}, \quad f \in \mathcal{F}. \tag{1.5.19}$$

Theorem 1.5.2 (DH1995:2.5 [?]). *Under the condition (d1) for $\{X_j\}$, the following are equivalent:*

(i) *N_n converges in distribution to some $N \neq o$.*

(ii) *For some finite positive constant γ, $k_n P[\vee_{k=1}^{r_n} |X_k| > a_n x] \to \gamma x^{-\alpha}$, $x > 0$, and for some probability measure $\mathcal{O}$ on $\tilde{M}$, $P[\sum_{j=1}^{r_n} \delta_{X_j/(\vee_1^{r_n}|X_k|)}] \in \cdot | \vee_1^{r_n} |X_k|| > a_n x] \to_w \mathcal{O}$, $x > 0$.*

In this case N is infinitely divisible with canonical measure λ confined to M_0 and satisfying

$$\lambda \circ \Omega^{-1} = \nu \times \mathcal{O}, \tag{1.5.20}$$

where $\nu(dy) = \gamma\alpha y^{-\alpha-1}\, dy$.

Theorem 1.5.3 (DH1995:2.6 [?]). *Suppose that $N_n \to_d N$ and N has the representation given by Theorem 1.5.1. Then $\gamma \sum_{i=1}^\infty E|Q_i|^\alpha \leq 1$, where $\sum_{i=1}^\infty \delta_{Q_i} \sim \mathcal{O}$. The equality holds if $\{N_n([-1,1]^c)\}_{n=1}^\infty\}$ is uniformly integrable.*

Theorem 1.5.4 (DH1995:2.7 [?]). *Suppose that $\{X_j\}$ is a stationary sequence of random variables for which all finite-dimensional distributions are jointly regularly varying with index $\alpha > 0$. To be specific, let $\boldsymbol{\theta}^{(m)} = (\theta_i^{(m)}, |i| \leq m)$ be the ransom vector $\boldsymbol{\theta}$ that appears* *in the definition of joint regular variation of X_i, $|i| \le m$. Assume that the condition (d1) holds for $\{X_j\}$ and that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\bigvee_{m\le|i|\le r_n} |X_i| > ta_n \middle| |X_0| > ta_n \right] = 0, \quad t > 0, \tag{1.5.21}$$

where a_n is defined above. Then the limit

$$\gamma := \lim_{m\to\infty} \frac{E(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha)_+}{E|\theta_0^{(m)}|^\alpha} \tag{1.5.22}$$

exists. If $\gamma = 0$ then $N_n \to_d o$; if $\gamma > 0$, then N_n converges in distribution to some N, where, using the representation $\lambda \circ \Omega^{-1} = \nu \times \mathcal{O}$ described in Theorem 1.5.1, $\nu(dy) = \gamma\alpha y^{-\alpha-1}\, dy$ and $\mathcal{O}$ is the weak limit of

$$\lim_{m\to\infty} \frac{E(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha)_+ I(\sum_{|i|\le m} \delta_{\theta_i^{(m)}} \in \cdot)}{E(|\theta_0^{(m)}|^\alpha - \vee_{j=1}^m |\theta_j^{(m)}|^\alpha)_+} \tag{1.5.23}$$

as $m \to \infty$, which exists.

Theorem 1.5.5 (DH1995:3.1 [?]). *Let $\{X_j\}$ be a strictly stationary sequence satisfying (c1) and*

Theorem 1.5.6 (1986:3.3 [4]). *Let $\{ Z_t \}$ be iid satisfying* (1.5.1) *and* (1.5.2) *with* $0 < \alpha < 2$ *and* $E|Z_1|^\alpha = \infty$. *Then, if a_n and $\tilde{a}_n$ are given by* (1.5.3) *and* (1.5.13),

$$(a_n^{-2} \sum_{t=1}^n Z_t^2, \tilde{a}_n^{-1} \sum_{t=1}^n (Z_t Z_{t+1} - \mu_n), \ldots, \tilde{a}_n^{-1} \sum_{t=1}^n (Z_t Z_{t+h} - \mu_n)) \Rightarrow (S_0, S_1, \ldots, S_h), \tag{1.5.24}$$

where $\mu_n = EZ_1Z_2 1_{[\,|Z_1Z_2| \le \tilde{a}_n\,]}$ and $S_0, S_1, \ldots, S_h$ are independent stable random variables; S_0 is positive with index $\alpha/2$ and $S_1, S_2, \ldots, S_h$ are identically distributed with index α.

Theorem 1.5.7. *Suppose $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ where $\{c_j\}$ satisfies* (1.5.15) *and $\{Z_t\}$ satisfies* (1.5.1) *and* (1.5.2), *and* $E|Z_1|^\alpha = \infty$, $0 < \alpha < 2$. *If a_n and $\tilde{a}_n$ are given by* (1.5.3) *and* (1.5.13), *then for any positive integer l,*

$$(\tilde{a}_n^{-1} a_n^2 (\hat{\rho}(h) - \rho(h) - d_{h,n}/C(0)), 1 \le h \le l) \Rightarrow (Y_1, Y_2, \ldots, Y_l) \tag{1.5.25}$$

in $\mathbb{R}^l$, where

$$\begin{aligned} d_{h,n} &= \sum_{j=1}^{\infty} (\rho(h+j) + \rho(h-j) - 2\rho(h)\rho(h)) \sum_i c_i^2 EZ_1Z_2 1_{[\,|Z_1Z_2| \le \tilde{a}_n\,]}, \\ Y_h &= \sum_{j=1}^{\infty} (\rho(h+j) + \rho(h-j) - 2\rho(j)\rho(h) S_j/S_0), \end{aligned}$$

and $S_0, S_1, S_2, \ldots$ are independent stable random variables as described in Theorem 3.3. In addition, if either

1. $0 < \alpha < 1$, *or*

2. $\alpha = 1$ *and the distribution of* Z_t *is symmetric, or*

3. $1 < \alpha < 2$ *and* $EZ_1 = 0$,

then (1.5.25) *holds with* $d_{h,n} = 0$, $h = 1, \ldots, l$, *and a location change in the* S_j*'s,* $j \geq 1$.

Theorem 1.5.8 (1985:4.2 [3]). *Let* $\sum_{k=1}^{\infty} \epsilon_{j_k}$ *be a PRM(*λ*) on* $\mathbb{R}\backslash\{0\}$ *with*

$$\lambda(dx) = \alpha p x^{-\alpha-1} 1_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)dx, \quad 0 < \alpha < 2. \tag{1.5.26}$$

Suppose (1.5.1) - (1.5.12), (1.5.6), (1.5.14) *hold with* $0 < \alpha < 2$. *Then for every non-negative integer* l*, as* $n \to \infty$*:*

1.

$$(n/a_n^2)(\hat{\gamma}(0), \hat{\gamma}(1), \cdots, \hat{\gamma}(l)) \Rightarrow \sum_{i=1}^{\infty} j_i^2 (\sum_{j=0}^{\infty} c_j^2, \sum_{j=0}^{\infty} c_j c_{j+1}, \cdot, \sum_{j=0}^{\infty} c_j c_{j+l}) \tag{1.5.27}$$

and

2.

$$\hat{\rho}(l) \to \rho(l) = \frac{\sum_{j=0}^{\infty} c_j c_{j+l}}{\sum_{j=0}^{\infty} c_j^2} \quad \textit{in probability.} \tag{1.5.28}$$

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